MATH 54 - TOPOLOGY
SUMMER 2015
TAKE-HOME EXAMINATION

ELEMENTS OF SOLUTION

Problem 1

The purpose of this problem is to study the connected components of $\mathbb{R}^\omega$ in various topologies. In what follows, B and U respectively denote the sets of bounded and unbounded sequences. Note that $\mathbb{R}^\omega = B \sqcup U$.

The sequence whose terms are constantly equal to 0 is denoted by $\mathbf{0}$. The connected component of x is denoted by C_x.

Finally, if x is an element in $\mathbb{R}^\omega$ and A a subset of $\mathbb{R}^\omega$, we denote by $x + A$ the set of A-translates of x, that is

$$x + A = \{x + a \ , \ a \in A\}.$$

1. Determine the connected components of $\mathbb{R}^\omega$ in the product topology.

For x, y in $\mathbb{R}^\omega$, the function

$$s_{x,y} : t \longmapsto (1-t)x + ty$$

is continuous from $[0,1]$ to $\mathbb{R}^\omega$ equipped with the box topology, because each component map s_{x_n,y_n} is polynomial hence continuous from $[0,1]$ to $\mathbb{R}$. Since each $s_{x,y}(t)$ is a real-valued sequence, it follows that $\mathbb{R}^\omega$ is convex, hence (path) connected.

2. Consider $\mathbb{R}^\omega$ equipped with the uniform topology.

(a) **Prove that x is in the same connected component as $\mathbf{0}$ if and only if x is bounded.**

We know that B is closed in $\mathbb{R}^\omega$ for the uniform topology. A similar argument shows that U is also closed, so that they constitute a separation of $\mathbb{R}^\omega$ in this topology. Therefore, since the zero sequence is bounded, we see that $C_{\mathbf{0}} \subset B$. Furthermore, for $x \in B$, the function $s_{\mathbf{0},x}$ satisfies

$$|s_{\mathbf{0},x}(t)| \leq t \cdot \sup_{n \geq 1} |x_n|$$

for every $t \in [0,1]$. It follows that it is continuous and that every $s_{\mathbf{0},x}(t)$ is bounded so B is connected in the uniform topology. Therefore, $C_{\mathbf{0}} = B$.

(b) **Deduce a necessary and sufficient condition for x and y in $\mathbb{R}^\omega$ to lie in the same connected component for the uniform topology.**

For x fixed in $\mathbb{R}^\omega$, the map $y \longmapsto y - x$ is a homeomorphism from $\mathbb{R}^\omega$ onto itself, which sends x to $\mathbf{0}$. It follows that x and y are in the same connected component if and only if $x - y \in C_{\mathbf{0}}$. In other words, $C_x = x + B$.

3. Consider $\mathbb{R}^\omega$ equipped with the box topology.

(a) **Let $x, y \in \mathbb{R}^\omega$ be such that $x - y \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$. Prove that there exists a homeomorphism $\varphi : \mathbb{R}^\omega \longrightarrow \mathbb{R}^\omega$ such that $(\varphi(x)_n)_{n\in\mathbb{Z}_+}$ is a bounded sequence and $(\varphi(y)_n)_{n\in\mathbb{Z}_+}$ is unbounded.**

Let us prove the following:
Lemma. Let $(\alpha_n)_{n\in\mathbb{Z}_+}$ and $(\beta_n)_{n\in\mathbb{Z}_+}$ be fixed sequences of real numbers such that $\alpha_n \neq 0$ for all n. Then the map

$$\begin{array}{cccc} \varphi : & \mathbb{R}^\omega & \longrightarrow & \mathbb{R}^\omega \\ & (u_n)_{n\in\mathbb{Z}_+} & \longmapsto & (\alpha_n u_n + \beta_n)_{n\in\mathbb{Z}_+} \end{array}.$$

is a homeomorphism in the box topology.

Proof. Since φ is bijective with inverse $(u_n)_{n\in\mathbb{Z}_+} \longmapsto \left(\frac{u_n}{\alpha_n} - \beta_n\right)_{n\in\mathbb{Z}_+}$ it suffices to prove that every map from $\mathbb{R}^\omega$ to itself with (non-constant) affine components is continuous. Let $W = \prod_{n\geq 1}(s_n, t_n)$ be a basis element for the box topology. Then,

$$\varphi^{-1}(W) = \prod_{n\geq 1}\left(\frac{s_n}{\alpha_n} - \beta_n, \frac{t_n}{\alpha_n} - \beta_n\right)$$

is also an open box, hence open. Therefore φ is continuous.

Now, with x and y fixed in $\mathbb{R}^\omega$ such that $x - y \notin \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, consider the map φ defined by

$$\varphi(u)_n = \begin{cases} u_n - x_n & \text{if } x_n = y_n \\ e^n \cdot \dfrac{u_n - x_n}{y_n - x_n} & \text{if } x_n \neq y_n \end{cases}$$

for $u \in \mathbb{R}^\omega$. Then φ is a homeomorphism of $\mathbb{R}^\omega$ by the lemma and $\varphi(x) = \mathbf{0} \in B$ while $\varphi(y)$ has an exponentially growing subsequence, hence $\varphi(y) \in U$.

(b) **Deduce a necessary and sufficient condition for x and y in $\mathbb{R}^\omega$ to lie in the same connected component for the box topology.**

Let $x, y \in \mathbb{R}^\omega$. Since $B \sqcup U$ is a separation of $\mathbb{R}^\omega$ in the box topology, any homeomorphism φ of $\mathbb{R}^\omega$ must satisfy $\varphi(C_x) \subset B$ or $\varphi(C_x) \subset U$. By the previous question, if follows that $y \in C_x$ implies $x - y \in \mathbb{R}^\infty$. In other words, $C_x \subset x + \mathbb{R}^\infty$.

The converse inclusion follows from a convexity argument. Assume that $x - y \in \mathbb{R}^\infty$. Then, $s_{x,y}(t) = x + t(y - x) \in x + \mathbb{R}^\infty$ for every $t \in [0, 1]$.

Let us prove that this map is continuous between $\mathbb{R}$ and $\mathbb{R}^\omega$ equipped with the box topology. Let $W = \prod_{n\geq 1} I_n$ with I_n an open interval of $\mathbb{R}$ for every $n \in \mathbb{Z}_+$. Then, for every n, the set $J_n = \{t \in [0,1]\ ,\ x_n + t(y_n - x_n)\}$ is

- empty or equal to $[0,1]$ if $x_n = y_n$;
- an open interval of $[0,1]$ if $x_n \neq y_n$.

Since the latter occurs only for finitely many values of n, the inverse image of W under $s_{x,y}$, which is $\bigcap_{n\geq 1} I_n$ is either empty or a finite intersection of open sets, hence open. This proves that $s_{x,y}$ is a continuous path, so that $x + \mathbb{R}^\infty$ is path connected, hence equal to C_x.

Problem 2

Let F be a functor between categories $\mathcal{C}$ and $\mathcal{C}'$. A functor $G : \mathcal{C}' \longrightarrow \mathcal{C}$ is said to be a *left adjoint* for F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(G(X), Y) \cong \mathrm{Hom}_{\mathcal{C}'}(X, F(Y))$$

for all objects $X \in \underline{\mathcal{C}}'$ and $Y \in \underline{\mathcal{C}}$. Similarly, G is called a *right adjoint* for F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{C}'}(F(X), Y)$$

for all objects $X \in \underline{\mathcal{C}}$ and $Y \in \underline{\mathcal{C}}'$.

Recall that the forgetful functor $\mathbb{F} : \mathbf{Top} \longrightarrow \mathbf{Set}$ is defined by

- **$\mathbb{F}((X, \mathcal{T})) = X$ for any set X equipped with a topology $\mathcal{T}$;**
- **$\mathbb{F}(f) = f$ for any continuous map $f : X \longrightarrow Y$.**

If X is a set, let $\mathbb{G}(X)$ denote the topological space obtained by endowing X with the trivial topology $\mathcal{T}_{\text{triv.}} = \{X, \emptyset\}$:

$$\mathbb{G}(X) = (X, \mathcal{T}_{\text{triv.}}).$$

If f is a map between sets, define in addition $\mathbb{G}(f) = f$.

This problem is a reformulation in the language of categories of the following basic properties of the trivial and discrete topologies:

(C1) If $(X, \mathcal{T})$ is a topological space, then any map $(X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}_{\text{triv.}})$ is continuous.

(C2) If $(Y, \mathcal{T})$ is a topological space, then any map $(X, \mathcal{T}_{\text{disc.}}) \longrightarrow (Y, \mathcal{T})$ is continuous.

1. Verify that $\mathbb{G}$ is a functor.

At the level of objects, $\mathbb{G}$ sends sets to topological spaces. To verify functoriality, it suffices to check that if f is a morphism in **Set**, then $\mathbb{G}(f)$ is a morphism in **Top** and compatibility with compositions in each categroy. Let X and Y be objects in **Set**, and $f \in \mathrm{Hom}(X, Y)$ that is, f is a map between sets X and Y. Then $\mathbb{G}(f) = f$ by definition and the condition

$$\mathbb{G}(f) \in \mathrm{Hom}(\mathbb{G}(X), \mathbb{G}(Y))$$

is equivalent to f being continuous between $(X, \mathcal{T}_{\text{triv.}})$ and $(Y, \mathcal{T}_{\text{triv.}})$, which follows from (C1). The composition relation is immediate as $\mathbb{G}(gf) = gf = \mathbb{G}(g)\mathbb{G}(f)$. Finally, $\mathbb{G}(\mathrm{Id}_X) = \{x \mapsto x\} = \mathrm{Id}_{\mathbb{G}(X)}$.

2. Prove that $\mathbb{G}$ is a right adjoint to $\mathbb{F}$.

To prove that $\mathbb{G}$ is a right adjoint to $\mathbb{F}$, we need to compare

$$\mathrm{Hom}_{\mathbf{Top}}((X, \mathcal{T}), \mathbb{G}(Y))$$

with

$$\mathrm{Hom}_{\mathbf{Set}}(\mathbb{F}\left((X, \mathcal{T})\right), Y)$$

for any topological space $(X, \mathcal{T})$ and every set Y. By definition, $\mathrm{Hom}_{\mathbf{Top}}((X, \mathcal{T}), \mathbb{G}(Y))$ is the set of continuous maps from $(X, \mathcal{T})$ to $(Y, \mathcal{T}_{\text{triv.}})$.

On the other hand, $\mathrm{Hom}_{\mathbf{Set}}(\mathbb{F}\left((X, \mathcal{T})\right), Y)$ consists of all maps from X to Y. Therefore, it follows from (C1) that every element of $\mathrm{Hom}_{\mathbf{Set}}(\mathbb{F}\left((X, \mathcal{T})\right), Y)$ can be seen as an element of $\mathrm{Hom}_{\mathbf{Top}}((X, \mathcal{T}), \mathbb{G}(Y))$. In other words, the natural isomorphism realizing the adjunction is the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{Set}}(\mathbb{F}\left((X, \mathcal{T})\right), Y) & \longrightarrow & \mathrm{Hom}_{\mathbf{Top}}((X, \mathcal{T}), \mathbb{G}(Y)) \\ f & \longmapsto & f \end{array}$$

3. Find a left adjoint for $\mathbb{F}$.

Similar arguments and (C2) imply that $\mathbb{H}$ defined on **Set** by $\mathbb{H}(X) = (X, \mathcal{T}_{\text{disc.}})$ and $\mathbb{H}(f) = f$ is a functorial and a left adjoint for $\mathbb{F}$.