MATH 54 - TOPOLOGY
SUMMER 2015
TAKE-HOME EXAMINATION

DUE MONDAY AUGUST 17

This is an individual assignment. You may use the text and class notes but no other source or outside help.

Problem 1

1. Determine the connected components of $\mathbb{R}^\omega$ in the product topology.

2. Consider $\mathbb{R}^\omega$ equipped with the uniform topology.

(a) Prove that x is in the same connected component as $\mathbf{0}$ if and only if x is bounded.

(b) Deduce a necessary and sufficient condition for x and y in $\mathbb{R}^\omega$ to lie in the same connected component for the uniform topology.

3. Consider $\mathbb{R}^\omega$ equipped with the box topology.

(a) Let $x, y \in \mathbb{R}^\omega$ be such that $x - y \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$.
Prove that there exists a homeomorphism

$$\varphi : \mathbb{R}^\omega \longrightarrow \mathbb{R}^\omega$$

such that $(\varphi(x)_n)_{n\in\mathbb{Z}_+}$ is a bounded sequence and $(\varphi(y)_n)_{n\in\mathbb{Z}_+}$ is unbounded.

Hint: given $u \in \mathbb{R}^\omega$, it might be helpful to consider the sequence v defined by

$$v_n = \begin{cases} u_n - x_n & \text{if } x_n = y_n \\ \frac{u_n - x_n}{y_n - x_n} & \text{if } x_n \neq y_n \end{cases} .$$

(b) Deduce a necessary and sufficient condition for x and y in $\mathbb{R}^\omega$ to lie in the same connected component for the box topology.

Problem 2

Let F be a functor between categories $\mathcal{C}$ and $\mathcal{C}'$. A functor $G : \mathcal{C}' \longrightarrow \mathcal{C}$ is said to be a *left adjoint* for F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(G(X), Y) \cong \mathrm{Hom}_{\mathcal{C}'}(X, F(Y))$$

for all objects $X \in \underline{\mathcal{C}}'$ and $Y \in \underline{\mathcal{C}}$. Similarly, G is called a *right adjoint* for F if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{C}'}(F(X), Y)$$

for all objects $X \in \underline{\mathcal{C}}$ and $Y \in \underline{\mathcal{C}}'$.

Recall that the forgetful functor $\mathbb{F} : \mathbf{Top} \longrightarrow \mathbf{Set}$ is defined by

- $\mathbb{F}\left((X, \mathcal{T})\right) = X$ for any set X equipped with a topology $\mathcal{T}$;
- $\mathbb{F}(f) = f$ for any continuous map $f : X \longrightarrow Y$.

If X is a set, let $\mathbb{G}(X)$ denote the topological space obtained by endowing X with the trivial topology $\mathcal{T}_{\text{triv.}} = \{X, \emptyset\}$:

$$\mathbb{G}(X) = (X, \mathcal{T}_{\text{triv.}}).$$

If f is a map between sets, define in addition $\mathbb{G}(f) = f$.

1. Verify that $\mathbb{G}$ is a functor.

2. Prove that $\mathbb{G}$ is a right adjoint to $\mathbb{F}$.

3. Find a left adjoint for $\mathbb{F}$.