MATH 54 - TOPOLOGY
SUMMER 2015
MIDTERM 2

ELEMENTS OF SOLUTION

Problem 1

1. Show that a topological space is T_1 if and only if for any pair of distinct points, each has a neighborhood that does not contain the other.

Let $x \neq y$ be elements of X, assumed T_1. Then $\{x\}$ is closed so $X \setminus \{x\}$ is a neighborhood of y that does not contain x. Similarly, $X \setminus \{y\}$ is a neighborhood of x that does not contain y. Conversely, assume that distinct points have neighborhoods that does not contain the other and let $x \in X$. Then if $y \neq x$, there is a neighborhood of y that does not contain x so $X \setminus \{x\}$ is open hence $\{x\}$ is closed.

2. Determine the interior and the boundary of the set

$$\Xi = \left\{(x,y) \in \mathbb{R}^2 \ , \ 0 \leq y < x^2+1\right\}$$

where $\mathbb{R}^2$ is equipped with its ordinary Euclidean topology.

$$\mathring{\Xi} = \left\{(x,y) \in \mathbb{R}^2 \ , \ 0 < y < x^2+1\right\}$$

$$\partial\Xi = \{y = 0\} \cup \{y = x^2+1\}$$

Problem 2

Let E be a set with a metric d and $\mathcal{T}_d$ the corresponding metric topology.

1. Prove that the map $d : (E, \mathcal{T}_d) \times (E, \mathcal{T}_d) \longrightarrow \mathbb{R}$ is continuous.

Let (a,b) be an arbitrary basis element for the topology on $\mathbb{R}$, with $b > 0$, so that $d^{-1}((a,b))$ is not empty. Let $(x,y) \in d^{-1}((a,b))$ and $d = d(x,y)$. Then, by the triangle inequality,

$$(p,q) \in B(x, \frac{b-d}{2}) \times B(y, \frac{b-d}{2}) \Rightarrow d(p,q) < b.$$

The triangle inequality also implies that $d(p,q) \geq d(x,y) - d(x,p) - d(v,y)$ so

$$(p,q) \in B(x, \frac{d-a}{2}) \times B(y, \frac{d-a}{2}) \Rightarrow a < d(p,q).$$

It follows that $B(x,r) \times B(y,r)$ with $r = \min\left\{\frac{b-d}{2}, \frac{d-a}{2}\right\}$ is a neighborhood of (x,y) contained in $d^{-1}((a,b))$, which is therefore open.

2. Let $\mathcal{T}$ be a topology on E, such that $d : (E, \mathcal{T}) \times (E, \mathcal{T}) \longrightarrow \mathbb{R}$ is continuous. Prove that $\mathcal{T}$ is finer than $\mathcal{T}_d$.

It suffices to prove that every ball $B(x, r)$ is open for $\mathcal{T}$. If $y \in B(x, r)$, then (x, y) belongs to $d^{-1}((-\infty, r))$, assumed open, so there exists a basis element $U \times V$ in $\mathcal{T} \times \mathcal{T}$ such that

$$(x, y) \in U \times V \subset d^{-1}((-\infty, r)).$$

In particular, V is a neighborhood of y. Moreover, if $z \in V$, then $(x, z) \in U \times V \subset d^{-1}((-\infty, r))$ so $d(x, z) < r$, which proves that $V \subset B(x, r)$, hence $B(x, r) \in \mathcal{T}$.

We have proved that the metric topology is the coarsest topology on E making d continuous.

Problem 3

We prove that the box topology on $\mathbb{R}^\omega$ is not metrizable.

1. Recall the definition of the box topology on $\mathbb{R}^\omega$.

It is the topology generated by the basis $\left\{\prod_{n\geq 1} U_n \text{ , } U_n \text{ open in } \mathbb{R}\right\}$.

Denote by 0 the sequence constantly equal to 0 and let

$$P = (0, +\infty)^\omega = \prod_{n\geq 1}(0, +\infty)$$

be the subset of positive sequences.

2. Verify that 0 belongs to $\bar{P}$.

Let $U = \prod_{n\geq 1} U_n$ be a neighborhood of $\mathbf{0}$. Then U_n is a neigborhood of 0 in $\mathbb{R}$ for every n. Therefore, U_n contains an interval (a_n, b_n) with $a_n < 0 < b_n$ for every n so the sequence $(b_n)_{n\geq 1}$ is an element of $U \cap P$. Every neighborhood of $\mathbf{0}$ meets P so $\mathbf{0} \in \bar{P}$.

3. Prove that no sequence $(p_n)_{n\geq 1} \in P^\omega$ converges to 0 in the box topology.

Let $({}^n u)_{n\geq 1}$ be a sequence of elements of P and consider the open box

$$\mathfrak{B} = \prod_{n\geq 1}(-{}^n u_n, {}^n u_n).$$

Then $\mathbf{0} \in \mathfrak{B}$, but no ${}^n u$ belongs to $\mathfrak{B}$, since the n^{th} term of ${}^n u$ lies outside the n^{th} interval in the product defining $\mathfrak{B}$.

4. Conclude.

In a metrizable space, closure points of are limits of sequences. Here, $\mathbf{0}$ is a closure point of P that is the limit of no sequence of elements of P. Therefore, the box topology on $\mathbb{R}^\omega$ is not metrizable.

Problem 4

1. Let X be a set.

(a) Recall the definition of the uniform topology on $\mathbb{R}^X$.

It is the metric topology associated with $\bar{\rho}(f,g) = \sup_{x\in X} \min\{|f(x)-g(x)|, 1\}$.

(b) Recall the definition of uniform convergence for a sequence in $\mathbb{R}^X$.

The sequence $(f_n)_{n\geq 1}$ converges uniformly to f in $\mathbb{R}^X$ if

$$\forall \varepsilon > 0 \ , \ \exists N_\varepsilon \in \mathbb{Z}_+ \ , \ \forall n \geq N_\varepsilon \ , \ \forall x \in X \ , \ |f_n(x) - f(x)| < \varepsilon.$$

2. Prove that a sequence in $\mathbb{R}^X$ converges uniformly if and only if it converges for $\mathcal{T}_\infty$.

Assume that f_n converges uniformly to f and let $0 < \varepsilon < 1$. Then for $n \geq N_{\frac{\varepsilon}{2}}$ and all $x \in X$,

$$\min\{|f(x) - g(x)|, 1\} = |f_n(x) - f(x)| < \frac{\varepsilon}{2},$$

so

$$\bar{\rho}(f_n, f) = \sup_{x\in X} \min\{|f_n(x) - f(x)|, 1\} \leq \frac{\varepsilon}{2} < \varepsilon,$$

which means that f_n converges to f in the uniform topology.

Conversely, assume that $\lim_{n\to\infty} \bar{\rho}(f_n, f) = 0$ and let $0 < \varepsilon < 1$. For n large enough, $\sup_{x\in X}\{|f_n(x) - f(x)| < \varepsilon$, so that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in X,$$

so f_n converges uniformly to f.

Problem 5

Consider the space $\mathbb{R}^\omega$ of real-valued sequences, equipped with the uniform topology.

1. Prove that the subset B of bounded sequences is closed.

It suffices to prove that a uniform limit of bounded sequences is bounded. Let $({}^n u)_{n\geq 1}$ be such that each $({}^n u_k)_{k\geq 1}$ is bounded:

$$|{}^n u_k| \leq M_n \quad \text{for all } k \geq 1$$

and assume that $\lim_{n\to\infty} {}^n u = u$, uniformly. Then there exists an integer n_0 such that

$$(\star) \qquad \forall n \geq n_0 \ , \ \sup_{k\geq 1} |{}^n u_k - u_k| < 1.$$

If u were unbounded, it would admit a subsequence u_{k_ℓ} such that $\lim_{\ell\to\infty} |u_{k_\ell}| = +\infty$. Since ${}^{n_0}u$ is bounded by M_{n_0}, this would imply that

$$\lim_{\ell\to\infty} |{}^{n_0} u_{k_\ell} - u_{k_\ell}| = +\infty,$$

which contradicts $(\star)$. Therefore, u must be bounded and B contains all its limit points.

2. Let $\mathbb{R}^\infty$ denote the subset of sequences with finitely many non-zero terms. Determine the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ for the uniform topology.

We will prove that $\overline{\mathbb{R}^\infty} = c_0(\mathbb{Z}_+)$, the set of sequences that converge to 0.

Let $({}^n u)_{n\geq 1}$ be a uniformly convergent sequence of elements of $\mathbb{R}^\infty$ and $u = \lim_{n\to\infty} {}^n u$. If u does not converge to 0, there exists some $\eta > 0$ such that

$$|u_k| > \eta$$

for arbitrarily large values of k. It follows that, for any $n \geq 1$,

$$|{}^n u - u_k| = |u_k| > \eta \quad \text{for some } k$$

since ${}^n u$ has only finitely many non-zero terms. This implies that $\sup_{k\geq 1} |{}^n u_k - u_k| \geq \eta$ for all $n \geq 1$, which contradicts the uniform convergence of $({}^n u)_{n\geq 1}$.

Conversely, any sequence u in $c_0(\mathbb{Z}_+)$ is the uniform limit of its truncations: let ${}^n u$ be the sequence defined by

$${}^n u_k = \begin{cases} u_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases} .$$

Then, ${}^n u \in c_0(\mathbb{Z}_+)$ and

$$\sup_{k\geq 1} |{}^n u_k - u_k| = \sup_{k>n} |u_k| \underset{n\to\infty}{\longrightarrow} 0$$

so ${}^n u$ converges uniformly to u.