MATH 54 - TOPOLOGY
SUMMER 2015
MIDTERM 2

DURATION: 2 HOURS

This exam consists of 5 independent problems. You may treat them in the order of your choosing, starting each problem on a new page.

Every claim you make must be **fully justified or quoted** as a result from the textbook.

Problem 1

1. Show that a topological space is T_1 if and only if for any pair of distinct points, each has a neighborhood that does not contain the other[1].

2. Determine the interior and the boundary of the set

$$\Xi = \left\{(x, y) \in \mathbb{R}^2 \ , \ 0 \leq y < x^2 + 1\right\}$$

where $\mathbb{R}^2$ is equipped with its ordinary Euclidean topology.

Problem 2

Let E be a set with a metric d and $\mathcal{T}_d$ the corresponding metric topology on E.

1. Prove that the map $d : (E, \mathcal{T}_d) \times (E, \mathcal{T}_d) \longrightarrow \mathbb{R}$ is continuous.

2. Let $\mathcal{T}$ be a topology on E, such that $d : (E, \mathcal{T}) \times (E, \mathcal{T}) \longrightarrow \mathbb{R}$ is continuous.
Prove that $\mathcal{T}$ is finer than $\mathcal{T}_d$.

Hint: it might be helpful to consider sets of the form $d^{-1}((-\infty, r))$ *for* $r > 0$.

[1] A topological space is said T_1 if all its singletons are closed.

Problem 3

The purpose of this problem is to prove that the box topology on $\mathbb{R}^\omega$ is not metrizable.

1. Recall the definition of the box topology on $\mathbb{R}^\omega$.

Denote by $\mathbf{0}$ the sequence constantly equal to 0 and let

$$P = (0, +\infty)^\omega = \prod_{n\geq 1}(0, +\infty)$$

be the subset of positive sequences.

2. Verify that $\mathbf{0}$ belongs to $\bar{P}$.

3. Prove that no sequence $(p_n)_{n\geq 1} \in P^\omega$ converges to $\mathbf{0}$ in the box topology.

4. Conclude.

Problem 4

1. Let X be a set.

(a) Recall the definition of the uniform topology on $\mathbb{R}^X$.

(b) Recall the definition of uniform convergence for a sequence of functions f_n in $\mathbb{R}^X$.

2. Prove that a sequence in $\mathbb{R}^X$ converges uniformly if and only if it converges for the uniform topology.

Problem 5

Consider the space $\mathbb{R}^\omega$ of real-valued sequences, equipped with the uniform topology.

1. Prove that the subset B of bounded sequences is closed in $\mathbb{R}^\omega$ for the uniform topology.

2. Let $\mathbb{R}^\infty$ denote the subset of sequences with finitely many non-zero terms.
Determine the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ for the uniform topology.