MATH 54 - TOPOLOGY
SUMMER 2015
MIDTERM 1

ELEMENTS OF SOLUTION

[M] refers to *Topology*, 2nd ed. by J. Munkres.

Problem 1

Let $\mathcal{T}$ be the family of subsets $\mathcal{U}$ of $\mathbb{Z}_+$ satisfying the following property:
If n is in $\mathcal{U}$, then any divisor of n belongs to $\mathcal{U}$.

1. Give two different examples of elements of $\mathcal{T}$ containing 24.

The prime factorization of 24 is $24 = 2^3 \cdot 3$ so divisors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24. Any element of $\mathcal{T}$ containing 24 must contain all of these. If we adjoin another number to this list, we must also include all of its divisors. For instance, any open set containing 10 will also contain 5. Examples are $\{1, 2, 3, 4, 5, 6, 8, 10, 12, 24\}$ or $\{1, 2, 3, 4, 6, 8, 12, 17, 24\}$.

2. Verify that $\mathcal{T}$ is a topology on $\mathbb{Z}_+$.

We verify the three axioms. (O1) The empty set trivially belongs to $\mathcal{T}$ and $\mathbb{Z}_+$ contains the divisors of any integer so it is open too.
(O2) Let $\{U_\alpha\}_{\alpha \in A}$ be a family of elements of $\mathcal{T}$ and $U = \bigcup_{\alpha \in A} U_\alpha$. Let $n \in U$. There exists $\alpha_0 \in A$ such that $n \in U_{\alpha_0}$ so all the divisors of n belong to U_{α_0} open, hence to U. This means that U is open by definition.
(O3) Let $\{U_i\}_{1 \leq i \leq p}$ be a family of elements of $\mathcal{T}$ and $U = \bigcap_{i=1}^{p} U_i$. Let $n \in U$. For every $1 \leq i \leq p$, the integer n belongs to U_i so every divisor of n belongs to all the U_i's, hence to U which is therefore open. This proves that $\mathcal{T}$ is a topology.

3. Is $\mathcal{T}$ the discrete topology?

Since 1 is a divisor of any integer, every non-empty open set must contain 1. Therefore, there exists subsets of $\mathbb{Z}_+$ that are not open and $\mathcal{T}$ is not discrete.

Problem 2

Let (E, d) be a metric space.

1. Recall the definition of the metric topology and prove that open balls form a basis for that topology.

A subset Ω of E is open if any point of Ω is contained in an open ball included in Ω. By [M, Lemma 13.2], this implies that balls form a basis for the metric topology.

2. Assume that ρ is a second metric on E such that, for every $x, y \in E$,

$$\frac{1}{2}d(x,y) \leq \rho(x,y) \leq 2d(x,y).$$

Compare the topologies generated by d and ρ.

Denote by $\mathcal{T}_d$ and $\mathcal{T}_\rho$ the topologies associated with the metrics d and ρ. We shall prove that $\mathcal{T}_\rho$ is finer than $\mathcal{T}_d$.
Let $B_d(a,r)$ be a ball and $x \in B_d(a,r)$. Since $B_d(a,r)$ is open for $\mathcal{T}_d$, there exists a radius $r' > 0$ such that $B_d(x,r') \subset B_d(a,r)$. The inequalities satisfied by d and ρ imply that

$$x \in B_\rho(x, \frac{r'}{2}) \subset B_d(x,r').$$

Indeed, if $\rho(x,y) < \frac{r'}{2}$, then $\frac{1}{2}d(x,y) < \frac{r'}{2}$ so $y \in B_d(x,r')$.
Lemma 13.3 in [M] then implies that $\mathcal{T}_d \subset \mathcal{T}_\rho$. The converse inclusion can be proved by the same argument, using the other inequality satisfied by ρ and d and we can conclude that $\mathcal{T}_d = \mathcal{T}_\rho$.

Problem 3

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X.

1. Verify that $\mathcal{T}_1 \cup \mathcal{T}_2$ is a subbasis for a topology.

Any $x \in X$ is included in an element of $\mathcal{T}_1$ (e.g X) hence of $\mathcal{T}_1 \cup \mathcal{T}_2$, which is therefore a subbasis for a topology.

From now on, $\mathcal{T}_1 \vee \mathcal{T}_2$ denotes the topology generated by $\mathcal{T}_1 \cup \mathcal{T}_2$.

2. Describe $\mathcal{T}_1 \vee \mathcal{T}_2$ when $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

If $\mathcal{T}_1 \subset \mathcal{T}_2$, then $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}_2$ so $\mathcal{T}_1 \vee \mathcal{T}_2$ is the topology generated by $\mathcal{T}_2$, that is, $\mathcal{T}_2$ itself.

3. Compare $\mathcal{T}_1 \vee \mathcal{T}_2$ with $\mathcal{T}_1$ and $\mathcal{T}_2$ in general.

By definition, $\mathcal{T}_1 \vee \mathcal{T}_2$ contains $\mathcal{T}_1$ and $\mathcal{T}_2$ so it is finer than both.

4. Let $\mathcal{T}$ be a finer topology than $\mathcal{T}_1$ and $\mathcal{T}_2$. Prove that $\mathcal{T}$ is finer than $\mathcal{T}_1 \vee \mathcal{T}_2$.

Any element of $\mathcal{T}_1 \vee \mathcal{T}_2$ is the union of subsets of X of the form $U = \bigcap_{i=1}^{n} U_i$ with $U_i \in \mathcal{T}_1 \cup \mathcal{T}_2$ for each i. Such a U is an intersection of elements of $\mathcal{T}_1$ and of $\mathcal{T}_2$, all of which belong to $\mathcal{T}$ assumed finer so U belongs to $\mathcal{T}$. Since $\mathcal{T}$ is stable under unions, it follows that $\mathcal{T}_1 \vee \mathcal{T}_2 \subset \mathcal{T}$. In other words, $\mathcal{T}_1 \vee \mathcal{T}_2$ is the coarsest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$

Problem 4

1. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$A = \left\{x\ ,\ \frac{1}{2} < |x| < 1\right\} = (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$ is open in $\mathbb{R}$ as the union of two basis elements. It is also open in Y by definition.

$B = \left\{x\ ,\ \frac{1}{2} < |x| \leq 1\right\} = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ is not open in $\mathbb{R}$ since no open interval containing 1 is included in B. It is open in Y as the intersection of Y with $(-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$, open in $\mathbb{R}$.

$C = \left\{x\ ,\ \frac{1}{2} \leq |x| < 1\right\} = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$ is not open in $\mathbb{R}$ since no open interval containing $\frac{1}{2}$ is included in C. It is not open in Y for the same reason: no intersection of Y with an open interval containing $\frac{1}{2}$ is included in C.

$D = \left\{x\ ,\ 0 < |x| < 1 \text{ and } \frac{1}{x} \in \mathbb{Z}_+\right\} = \left\{\frac{1}{n}\ ,\ n \in \mathbb{Z}_+\ ,\ n \geq 2\right\}$ is neither open in $\mathbb{R}$ nor in Y. The argument used for C can be used without modification.

2. Let $X = \mathbb{R}_\ell \times \mathbb{R}_u$ where $\mathbb{R}_\ell$ denotes the topology with basis consisting of all intervals of the form $[a, b)$ and $\mathbb{R}_u$ denotes the topology with basis consisting of all intervals of the form $(c, d]$. Describe the topology induced on the plane curve Γ with equation $y = e^x$.

A basis for the subspace topology on Γ is given by the sets $[a, b) \times (c, d] \cap \Gamma$. Singletons are of this form: $\{(x, e^x)\} = [x, x+1) \times (e^x - 1, e^x] \cap \Gamma$ so the induced topology is discrete.