Math 54 Summer 2015
Homework #7: connectedness and compactness - Elements of solution

(1) **Let U be an open connected subspace of $\mathbb{R}^2$ and $a \in U$.**

(a) **Prove that the set Γ_a of points $x \in U$ such that there is a path $\gamma : [0,1] \longrightarrow U$ with $\gamma(0) = a$ and $\gamma(1) = x$ is open and closed in U.**

Let x be an element of Γ_a, connected to a by a path γ, and $r > 0$ such that $B(x,r) \subset U$. Then for $y \in B(x,r)$, the map $\tilde{\gamma} : [0,1] \longrightarrow U$ defined by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2(1-t)x + (2t-1)y & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a continuous path joining a to y, so $B(x,r) \subset \Gamma_a$, which is therefore open.

To prove that it is closed, let $x \in U$ be an accumulation point of Γ_a and $r > 0$ such that $B(x,r) \subset U$. Since x is an accumulation point, there exists $y \neq x$ in $B(x,r) \cap \Gamma_a$. Then, as in the proof that Γ_a is open, one can concatenate a path from a to y and the segment from y to x to get a continuous path in U that connects a to x. It follows that $x \in \Gamma_a$, which means that Γ_a contains its accumulation points, hence is closed in U.

(b) **What can you conclude?**

The set of points that can be connected to a by a path is open and closed in U connected. Since it is not empty (it contains a) it is equal to U, which is therefore path connected. In other words connectedness and path connectedness are equivalent for open subsets of $\mathbb{R}^2$.

(2) **Let X be a topological space and $Y \subset X$ a connected subspace.**

(a) **Are $\mathring{Y}$ and ∂Y necessarily connected?**

The answer is negative in both cases. Let $Y_1 = L \cup R \subset \mathbb{R}^2$ be the union of the half plane $L = \{x \leq 0\}$ and the half-cone $R = \{x \geq 0\ ,\ |y| \leq x\}$. Then Y_1 is connected because both L and R are, and they intersect at the origin. On the other hand,

$$\mathring{Y}_1 = \{x < 0\} \sqcup \{x > 0\ ,\ |y| < x\}$$

is disconnected, as the two terms in the union are disjoint and open.

To see that a connected set need not have a connected boundary, it suffices to consider a closed interval of $\mathbb{R}$. Another example is that of a closed washer in $\mathbb{R}^2$: let $Y_2 = \{1 \leq x^2 + y^2 \leq 4\}$. It is homeomorphic to the rectangle $[1,2] \times [0, 2\pi)$ *via* polar coordinates, hence connected. However, its boundary consists of two disjoint circles, closed in $\mathbb{R}^2$:

$$\partial Y_2 = \{x^2 + y^2 = 1\} \sqcup \{x^2 + y^2 = 4\}$$

which are closed as inverse images of singletons under a continuous map.

(b) **Does the converse hold?**

No: in $\mathbb{R}$, consider the union of negative real number and positive rationals

$$Y_3 = (-\infty, 0) \cup \mathbb{Q}_+.$$

Then $\mathring{Y_3} = (-\infty, 0)$ and $\partial Y_3 = [0, +\infty]$ are connected, but Y_3 is disconnected as each rational is alone in its connected component.

(3) **Let (E, d) be a metric space.**

(a) **Prove that every compact subspace of E is closed and bounded.**

In a metric spaces[1], compact sets are closed [M. Th.26.3]. Moreover, assume that K is compact in E and that K can be covered by open balls of radius 1. By compactness, K can be covered by finitely many ball of radius 1, say N. Then the triangle inequality shows that $d(x, y) \leq 2N$, so K is bounded.

(b) **Give an example of metric space in which closed bounded sets are not necessarily compact.**

Let X be an infinite set, equipped with the discrete metric. Then X is closed and bounded for the corresponding metric topology. However, the cover given by all singletons, which are open, since the topology is discrete, does not have any finite subcover.

(4) This problem gives concrete descriptions of the Alexandrov compactifications of some locally compact spaces. The Alexandrov compactification is defined up to homeomorphism and it follows from the universal property that homeomorphic spaces have the same compactification.

(a) **Prove that the Alexandrov compactification of $\mathbb{R}$ is homeomorphic to the unit circle $\mathrm{S}^1 = \{(x, y) \in \mathbb{R}^2 \ , \ x^2 + y^2 = 1\}$.**

Observe that S^1 is compact as a closed and bounded subset of $\mathbb{R}^2$. Moreover, $\mathrm{S}^1 \setminus \{(-1, 0)\}$ can be parametrized by the map $r : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by

$$r(t) = \left(\frac{1 - t^2}{1 + t^2} \ , \ \frac{2t}{1 + t^2} \right).$$

This map is continuous because its components are rational functions with non-vanishing denominators, and takes values in S^1. Its inverse is

$$(x, y) \longmapsto \frac{y}{x + 1},$$

continuous on $\mathrm{S}^1 \setminus \{(-1, 0)\}$ for the same reason.

We have proved that $\mathbb{R} \cong \mathrm{S}^1 \setminus \{(-1, 0)\}$, which proves that $\tilde{\mathbb{R}} \cong \mathrm{S}^1$.

[1] This actually holds for Hausdorff spaces in general.

(b) **Verify that $\mathbb{Z}_+ \subset \mathbb{R}$ is a locally compact Hausdorff space.**

Subspaces inherit the Hausdorff property so $\mathbb{Z}_+$ is Haudorff because $\mathbb{R}$ is. Moreover, every finite subset of $\mathbb{Z}_+$ is compact for the subspace topology, which is discrete, $n \in \mathbb{Z}_+$ can be seen in $\{x\}$, compact and open.

(c) **Prove that the Alexandrov compactification of $\mathbb{Z}_+$ is homeomorphic to $\left\{\frac{1}{n}\,,\ n \in \mathbb{Z}_+\right\} \cup \{0\}$.**

Again, observe that $\left\{\frac{1}{n}\,,\ n \in \mathbb{Z}_+\right\} \cup \{0\}$ is closed and bounded in $\mathbb{R}$, hence compact. Moreover, the map $n \longmapsto \frac{1}{n}$ is a homeomorphism between $\mathbb{Z}_+$ and $\left\{\frac{1}{n}\,,\ n \in \mathbb{Z}_+\right\}$ so $\widetilde{\mathbb{Z}_+} \cong \left\{\frac{1}{n}\,,\ n \in \mathbb{Z}_+\right\} \cup \{0\}$.