Math 54 Summer 2015
Homework #7: connectedness and compactness

(1) Let U be an open connected subspace of $\mathbb{R}^2$ and $a \in U$.

  (a) Prove that the set of points $x \in U$ such that there is a path $\gamma : [0,1] \longrightarrow U$ with $\gamma(0) = a$ and $\gamma(1) = x$ is open and closed in U.

  (b) What can you conclude?

(2) Let X be a topological space and $Y \subset X$ a connected subspace.

  (a) Are $\mathring{Y}$ and ∂Y necessarily connected?

  (b) Does the converse hold?

(3) Let (E, d) be a metric space.

  (a) Prove that every compact subspace of E is closed and bounded.

  (b) Give an example of metric space in which closed bounded sets are not necessarily compact.

(4) (a) Prove that the Alexandrov compactification of $\mathbb{R}$ is homeomorphic to the unit circle

$$\mathrm{S}^1 = \{(x, y) \in \mathbb{R}^2 \ , \ x^2 + y^2 = 1\}.$$

  (b) Verify that $\mathbb{Z}_+ \subset \mathbb{R}$ is a locally compact Hausdorff space.

  (c) Prove that the Alexandrov compactification of $\mathbb{Z}_+$ is homeomorphic to

$$\left\{ \frac{1}{n} \ , \ n \in \mathbb{Z}_+ \right\} \cup \{0\}.$$