**Math 54 Summer 2015**
**Homework #6: metrizable spaces - Elements of solution**

(1) **Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. For $x \in \mathbb{R}^\omega$ and $0 < \varepsilon < 1$, let**

$$P(x,\varepsilon) = \prod_{n\in\mathbb{Z}_+} (x_n - \varepsilon, x_n + \varepsilon).$$

(a) **Compare $P(x,\varepsilon)$ with $B_{\bar{\rho}}(x,\varepsilon)$.**

With $0 < \varepsilon < 1$, the definition of $\bar{\rho}$ implies that

$$B_{\bar{\rho}}(x,\varepsilon) = \left\{ y \in \mathbb{R}^\omega \ , \ \sup_{n\geq 1} |x_n - y_n| < \varepsilon \right\}.$$

On the other hand, $P(x,\varepsilon) = \{y \in \mathbb{R}^\omega \ , \ |x_n - y_n| < \varepsilon \quad \text{for every } n \geq 1\}$ so the inclusion $B_{\bar{\rho}}(x,\varepsilon) \subset P(x,\varepsilon)$ holds by definition of the supremum. This inclusion is strict: the sequence $y$ defined by

$$y_n = x_n + \varepsilon - \frac{1}{n}$$

belongs to $P(x,\varepsilon)$ but $\bar{\rho}(x,y) = \varepsilon$ so $B_{\bar{\rho}}(x,\varepsilon) \subsetneq P(x,\varepsilon)$.

(b) **Is $P(x,\varepsilon)$ open in the uniform topology?**

No: we prove that $P(x,\varepsilon)$ contains no $\bar{\rho}$-ball centered at the element $y$ introduced above. For any $\eta > 0$, the sequence $z$ defined by $z_n = y_n + \frac{\eta}{2}$ satisfies $\bar{\rho}(y,z) < \eta$ hence belongs to $B_{\bar{\rho}}(y,\eta)$.
Observe that $x_n + \varepsilon - y_n < \dfrac{\eta}{2}$ for $n$ large enough, so

$$\begin{aligned} z_n - x_n &= y_n + \frac{\eta}{2} - x_n \\ &= \varepsilon - \frac{1}{n} + \frac{\eta}{2} \\ &> \varepsilon \end{aligned}$$

which means that $z \notin P(x,\varepsilon)$, which is therefore not a neighborhood of $y$, hence not open.

(c) **Show that $B_{\bar{\rho}}(x,\varepsilon) = \bigcup_{\delta<\varepsilon} P(x,\delta)$.**

Let $y \in P(x,\delta)$ with $\delta < \varepsilon$. Then, $\bar{\rho}(x,y) \leq \delta < \varepsilon$ so $y \in B_{\bar{\rho}}(x,\varepsilon)$, which proves that $B_{\bar{\rho}}(x,\varepsilon) \supset P(x,\delta)$. Since $\delta$ was arbitrary, we conclude that

$$B_{\bar{\rho}}(x,\varepsilon) \supset \bigcup_{\delta<\varepsilon} P(x,\delta).$$

Conversely, if $y \in B_{\bar{\rho}}(x,\varepsilon)$, then $y \in P(x,\delta_y)$ with $\delta_y = \bar{\rho}(x,y) < \varepsilon$, so

$$B_{\bar{\rho}}(x,\varepsilon) \subset \bigcup_{\delta<\varepsilon} P(x,\delta).$$

(2) **We denote by $\ell^2(\mathbb{Z}_+)$ the set of square-summable real-valued sequences:**

$$\ell^2(\mathbb{Z}_+) = \left\{ x = (x_n)_{n\in\mathbb{Z}_+} \in \mathbb{R}^\omega \quad , \quad \sum_{n\geq 1} x_n^2 \quad \textbf{converges} \right\},$$

**equipped with the metric**

$$d(x, y) = \left( \sum_{n\geq 1} (x_n - y_n)^2 \right)^{1/2}.$$

(a) **Compare the metric topology induced by $d$ on $\ell^2(\mathbb{Z}_+)$ with the restrictions of the box and uniform topologies from $\mathbb{R}^\omega$.**

The inclusion $\mathcal{T}_{\bar\rho} \subset \mathcal{T}_d$ follows from the observation that $B_d(x, r) \subset B_{\bar\rho}(x, r)$ for any $x \in \ell^2(\mathbb{Z}_+)$ and $r > 0$. Indeed, observe that

$$|x_k - y_k|^2 \leq \sum_{n\geq 1} |x_n - y_n|^2$$

for any fixed $k$, so that $\bar\rho(x, y) \leq d(x, y)$ for any $x, y \in \ell^2(\mathbb{Z}_+)$[1].

This inclusion is strict, as the following example shows. Denote by $\mathbf{0}$ the sequence that is constantly equal to 0. We will prove that $B_d(\mathbf{0}, 1)$, open in $\mathcal{T}_d$ by definition, is not open in the uniform topology. More precisely, no ball $B_{\bar\rho}(\mathbf{0}, r)$ with $r > 0$ is included in $B_d(\mathbf{0}, 1)$: let $n_0 > \left(\frac{2}{r}\right)^2$ and consider the sequence

$$\xi_n = \begin{cases} \frac{r}{2} & \text{if } n \leq n_0 \\ 0 & \text{if } n > n_0 \end{cases}.$$

Then $\bar\rho(\mathbf{0}, \xi) = \frac{r}{2} < r$ so $\xi \in B_{\bar\rho}(\mathbf{0}, r)$ but $d(\mathbf{0}, \xi) = \sqrt{n_0}\dfrac{r}{2} > 1$ so $\xi \notin B_d(\mathbf{0}, 1)$.

Next, we prove that $\mathcal{T}_d \subset \mathcal{T}_{\text{box}}$: let $x \in \ell^2(\mathbb{Z}_+)$, $r > 0$ and consider the box

$$\mathfrak{B} = \prod_{n\geq 1} (x_n - \frac{r}{2^n}, x_n + \frac{r}{2^n}).$$

Then $x \in \mathfrak{B} \subset B_d(x, r)$ since $y \in \mathfrak{B}$ implies $d(x, y)^2 \leq \displaystyle\sum_{n\geq 1} \frac{r^2}{4^n} = \frac{r^2}{3}$.

Again, the inclusion is strict: consider the open box

$$\mathfrak{B} = \prod_{n\geq 1} \left( -\frac{1}{n}, \frac{1}{n} \right).$$

Although $\mathbf{0} \in \mathfrak{B}$, no ball $B_d(\mathbf{0}, r)$ with $r > 0$ is included in $\mathfrak{B}$.

[1]Another approach to this result consists in proving that uniform convergence implies $\ell^2$ convergence and conclude by the sequential characterization of the closure.

Indeed, consider $n_0 > \frac{2}{r}$ and let $\eta$ be the sequence defined by

$$\eta_n = \begin{cases} \frac{r}{2} & \text{if } n = n_0 \\ 0 & \text{if } n \neq n_0 \end{cases}.$$

Then $x \notin \mathfrak{B}$ but $d(\mathbf{0}, x) = \frac{r}{2} < r$ so $x \in B_d(\mathbf{0}, r)$. Therefore $\mathfrak{B} \notin \mathcal{T}_d$.

We have proved that $\mathcal{T}_{\bar{\rho}} \subsetneq \mathcal{T}_d \subsetneq \mathcal{T}_{\text{box}}$.

(b) **Let $\mathbb{R}^\infty$ denote the subset of $\ell^2(\mathbb{Z}_+)$ consisting of sequences that have finitely many non-zero terms. Determine the closure of $\mathbb{R}^\infty$ in $\ell^2(\mathbb{Z}_+)$.**

We will prove that $\overline{\mathbb{R}^\infty}^{\ell^2(\mathbb{Z}_+)} = \ell^2(\mathbb{Z}_+)$, using the sequential characterization of the closure in a metric space. For any $x \in \ell^2(\mathbb{Z}_+)$, let ${}^n x$ be the sequence defined by

$${}^n x_k = \begin{cases} x_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}.$$

Then, $d({}^n x, x)^2 = \sum_{k>n} x_k^2$. This quantity converges to 0 as $n \to \infty$ because $x$ is square-summable, so $\lim\limits_{n\to\infty} {}^n x = x$ in $\ell^2(\mathbb{Z}_+)$.

(3) **Let $X$ be a topological space, $Y$ a metric space. Assume that $(f_n)_{n\geq 0}$ is a sequence of continuous functions that converges uniformly to $f : X \to Y$. Let $(x_n)_{n\geq 0}$ be a sequence in $X$ such that $\lim\limits_{n\to\infty} x_n = x$. Prove that**

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Let $\varepsilon > 0$ and observe that the triangle inequality implies that

$$d(f_n(x_n), f(x)) \leq \underbrace{d(f_n(x_n), f(x_n))}_{A_n} + \underbrace{d(f(x_n), f(x))}_{B_n}.$$

The convergence of the sequence is assumed uniform so there exists an integer $N_A$ such that $d(f_n(\xi), f(\xi)) < \dfrac{\varepsilon}{2}$ for any $n \geq N_A$ and any $\xi \in X$.
In particular, $A_n < \dfrac{\varepsilon}{2}$ for $n \geq N_A$.
Moreover, the Uniform Limit Theorem guarantees that $f$ is continuous. Therefore, there exists $N_B$ such that $B_n < \dfrac{\varepsilon}{2}$ for any $n \geq N_B$.
It follows that $d(f_n(x_n), f(x)) \leq \varepsilon$ for any $n$ greater than $\max(N_A, N_B)$.

(4) **Ultrametric spaces.**

**Let $X$ be a set equipped with a map $d : X \times X \longrightarrow \mathbb{R}$ such that**

(1) $d(x,y) \geq 0$ (3) $d(x,y) = 0 \Leftrightarrow x = y$

(2) $d(x,y) = d(y,x)$ (4) $d(x,z) \leq \max(d(x,y), d(y,z))$

(a) **Verify that $d$ is a distance.**

The only condition to check is the triangle inequality:

$$\begin{aligned} d(x,y) + d(y,z) &= \max(d(x,y), d(y,z)) + \min(d(x,y), d(y,z)) \\ &\geq \max(d(x,y), d(y,z)) \qquad \text{by (1)} \\ &\geq d(x,z) \qquad \text{by (4).} \end{aligned}$$

(b) **Let $B$ be an open ball for $d$. Prove that $B = B(y,r)$ for every element $y \in B$ for some $r > 0$.**

Let $x \in X$, $r > 0$ and $B = B(x,r)$. Let $y \in B$, that is, assume $d(x,y) < r$. Note that

$$d(x,z) < r \;\Leftrightarrow\; \max(\underbrace{d(x,y)}_{<r}, d(x,z)) < r,$$

which implies that $d(y,z) < r$ by (4). It follows that $B \subset B(y,r)$. The reverse inclusion follows from exchanging $x$ and $y$ in the previous argument.

**Every point in the ball is a center!**

(c) **Prove that closed balls are open and open balls are closed in the topology induced by $d$.**

Let $B$ be a closed ball, that is $B = B_c(x,r) = \{y \in X \;,\; d(x,y) \leq r\}$ for some $x \in X$ and $r > 0$. Let $y \in B$. The open ball $B\left(y, \frac{r}{2}\right)$ is a neighborhood of $y$ contained in $B$:

$$d(x,z) \underset{(4)}{\leq} \max(\underbrace{d(x,y)}_{\leq r} \;,\; \underbrace{d(y,z)}_{<r/2}) \leq r$$

for any $z \in B\left(y, \frac{r}{2}\right)$, so $B$ is open.

To prove that open balls are closed, we used the sequential characterization: let $(x_n)_{n\geq 1}$ be a sequence in $B(a,r)$ such that $\lim\limits_{n\to\infty} x_n = x$ for some $x \in X$. Since $d(x_n, x)$ can be made arbitrarily small, there exists an integer $n_0$ such that $d(x_{n_0}, x) < r$. The ultrametric property of $d$ implies that

$$d(a,x) \leq \max(d(a, x_{n_0}) \;,\; d(x_{n_0}, x)) < r,$$

so $x \in B(a,r)$, which is therefore sequentially closed.

**Note:** it might seem that a distance with such properties may not be useful in any reasonable circumstances or even not exist at all. It is easy to check that the discrete metric on any set is ultrametric. More interestingly the $p$-adic distances defined on $\mathbb{Q}$ are ultrametric, which gives $p$-adic analysis a very different flavor from that of real analysis.