Math 54 Summer 2015
Homework #6: metrizable spaces

(1) Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. For $x = (x_n)_{n\in\mathbb{Z}_+} \in \mathbb{R}^\omega$ and $0 < \varepsilon < 1$, let

$$P(x,\varepsilon) = \prod_{n\in\mathbb{Z}_+} (x_n - \varepsilon, x_n + \varepsilon).$$

(a) Compare $P(x,\varepsilon)$ with $B_{\bar{\rho}}(x,\varepsilon)$.
(b) Is $P(x,\varepsilon)$ open in the uniform topology?
(c) Show that $B_{\bar{\rho}}(x,\varepsilon) = \bigcup_{\delta<\varepsilon} P(x,\delta)$.

(2) We denote by $\ell^2(\mathbb{Z}_+)$ the set of square-summable real-valued sequences, that is,

$$\ell^2(\mathbb{Z}_+) = \left\{ x = (x_n)_{n\in\mathbb{Z}_+} \in \mathbb{R}^\omega \quad , \quad \sum_{n\geq 1} x_n^2 \quad \text{converges} \right\}.$$

We admit that the formula

$$d(x,y) = \left(\sum_{n\geq 1} (x_n - y_n)^2 \right)^{1/2}$$

defines a metric on $\ell^2(\mathbb{Z}_+)$.

(a) Compare the metric topology induced by d on $\ell^2(\mathbb{Z}_+)$ with the restrictions of the box and uniform topologies from $\mathbb{R}^\omega$.
(b) Let $\mathbb{R}^\infty$ denote the subset of $\ell^2(\mathbb{Z}_+)$ consisting of sequences that have finitely many non-zero terms. Determine the closure of $\mathbb{R}^\infty$ in $\ell^2(\mathbb{Z}_+)$.

(3) Let X be a topological space, Y a metric space and assume that $(f_n)_{n\geq 0}$ is a sequence of continuous functions that converges uniformly to $f : X \longrightarrow Y$.
Let $(x_n)_{n\geq 0}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

(4) **Ultrametric spaces** (*non-mandatory*).

Let X be a set equipped with a map $d : X \times X \longrightarrow \mathbb{R}$ such that for all $x, y, z \in X$,

(1) $d(x, y) \geq 0$
(2) $d(x, y) = d(y, x)$
(3) $d(x, y) = 0 \Leftrightarrow x = y$
(4) $d(x, z) \leq \max(d(x, y), d(y, z))$

(a) Prove that d is a distance.
(b) Let B be an open ball for d. Prove that $B = B(y, r)$ for every element $y \in B$ for some $r > 0$.
(c) Prove that closed balls are open and open balls are closed in the topology induced by d.