Math 54 Summer 2015
Homework #5: continuous maps, the product topology - Elements of solution

(1) (a) **Consider $\mathbb{Z}_+$ equipped with the topology in which open sets are the subsets U such that if n is in U, then any divisor of n belongs to U. Give a necessary and sufficient condition for a function $f : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ to be continuous.**

Assume f continuous. For $n \in \mathbb{Z}_+$, let U_n be the open set of all divisors of n. Let $a \in f^{-1}(U_n)$, assumed non-empty. Since f is continuous, $f^{-1}(U_n)$ is open, hence contains all the divisors of a. In other words, if $b|a$, then $f(b) \in U_n$, that is, $f(b)|n$. A necessary condition for continuity is therefore that f preserve divisibility:

$$b|a \Rightarrow f(b)|f(a).$$

Let us prove that the condition is also sufficient. Assume that $f(b)|f(a)$ whenever $b|a$ and let U be open in $\mathbb{Z}_+$. If $f^{-1}(U) =$, it is open. Otherwise, let $a \in f^{-1}(U)$. To prove that $f^{-1}(U)$ is open, it suffices to show that it contains all the divisors of a. The property of f implies that $f(b)|f(a)$ for every such divisor b and, U being open, this implies that $f(b) \in U$, that is, $b \in f^{-1}(U)$.

(b) **Let $\chi_{\mathbb{Q}}$ be the indicator of $\mathbb{Q}$. Prove that the map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\varphi(x) = x \cdot \chi_{\mathbb{Q}}(x)$ is continuous at exactly one point.**

We shall prove that φ is continuous at 0 and discontinuous everywhere else. Note that $|\varphi(x)| \leq |x|$ for every $x \in \mathbb{R}$. In particular, let $\varepsilon > 0$ and $\delta = \varepsilon$. Then $|x| < \delta$ implies $|\varphi(x)| < \varepsilon$, so φ is continuous at 0.
Now observe that φ is odd and let $a > 0$ be a positive number. Then,

$$\varphi^{-1}(\left(\frac{a}{2}, \frac{3a}{2}\right)) = \left(\frac{a}{2}, \frac{3a}{2}\right) \cap \mathbb{Q}$$

which is not open as no subset of $\mathbb{Q}$ can contain an open interval of $\mathbb{R}$. Therefore, φ is not continuous at a.

(2) **Let X and Y be topological spaces. If A is a subset of either, we denote by A' the sets of accumulation points of A and by ∂A its boundary. Let $f : X \longrightarrow Y$ be a map. Determine the implications between the following statements.**

(i) f **is continuous.**

(ii) $f(A') \subset (f(A))'$ **for any** $A \subset X$**.**

(iii) $\partial(f^{-1}(B)) \subset f^{-1}(\partial B)$ **for any** $B \subset Y$**.**

Considering a constant function $\mathbb{R} \longrightarrow \mathbb{R}$ shows (i)$\not\Rightarrow$(ii). However, the converse is true: let A be a subset of X and $x \in \bar{A} = A \cup A'$.

- If $x \in A$, then $f(x) \in f(A) \subset \overline{f(A)}$.
- If $x \in A'$, then (ii) implies that $f(x) \in (f(A))' \subset \overline{f(A)}$.

Therefore, $f(\bar{A}) \subset \overline{f(A)}$ for any $A \subset X$ so f is continuous by [M. Th. 18.1].

Let us prove that (i) $\Rightarrow$ (iii). Assume f continuous, let $B \subset Y$ be a subset and $x \in \partial(f^{-1}(B))$. If $x \notin f^{-1}(\partial B)$, there are two possibilities.

Case 1: $f(x) \in \mathring{B}$. Then $x \in f^{-1}(\mathring{B})$, open by continuity of f. Since $f^{-1}(\mathring{B}) \subset f^{-1}(B)$, it follows that x is an interior point of $f^{-1}(B)$, which is a contradiction.

Case 2: $f(x) \in Y \setminus \bar{B}$. Then $x \in f^{-1}(Y \setminus \bar{B})$, open by (i). In particular, there is a neighborhood U of x such that $U \subset f^{-1}(Y \setminus \bar{B})$. Since $x \in \partial(f^{-1}(B))$, it follows that there exists some y in U such that $f(y) \in B$, which contradicts the assumption that $f(x) \in Y \setminus \bar{B}$.

Altogether, this proves that $x \in f^{-1}(\partial B)$, hence the inclusion of (iii).

To establish the converse, we rely on the following characterization of continuity:

Lemma: f is continuous if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for every $B \subset Y$.

Proof of the lemma: if f is continuous, the inverse image of the closed set $\bar{B}$ is a closed set that contains $f^{-1}(B)$ hence its closure. Conversely, if B is closed, the condition becomes $\overline{f^{-1}(B)} \subset f^{-1}(B)$. The reverse inclusion holds by definition of the closure, so $\overline{f^{-1}(B)} = f^{-1}(B)$, hence $f^{-1}(B)$ is closed and f is continuous.

If f is discontinuous, the lemma implies the existence of some $B \subset Y$ such that $\overline{f^{-1}(B)} \not\subset f^{-1}(\bar{B})$. Let x be an element of $\overline{f^{-1}(B)}$ such that $f(x) \notin \bar{B}$, hence $f(x) \notin B$. Since $\overline{f^{-1}(B)} = f^{-1}(B) \cup \partial f^{-1}(B)$, it follows that $x \in \partial f^{-1}(B)$. The fact that $f(x) \notin \bar{B}$ implies that $f(x) \notin \bar{B} \setminus \mathring{B} = \partial B$, that is

$$\partial(f^{-1}(B)) \not\subset f^{-1}(\partial B)$$

and we have proved the contrapositive of (iii) $\Rightarrow$ (i).

To sum up, conditions (i) and (iii) are equivalent and they are implied by (ii), but the converse does not hold:

$$\text{(ii)} \Rightarrow \text{(i)} \Leftrightarrow \text{(iii)}.$$

(3) **Let X and Y be topological spaces, and assume Y Hausdorff. Let A be a subset of X and f_1, f_2 continuous maps from the closure $\bar{A}$ to Y. Prove that if f_1 and f_2 restrict to the same function $f : A \to Y$, then $f_1 = f_2$.**

We argue by contradiction: if $f_1 \neq f_2$, there exists $x \in \bar{A}$ such that $f_1(x) \neq f_2(x)$ and $x \notin A$. Since Y is Hausdorff, there exist disjoint neighborhoods V_1 of $f_1(x)$ and V_2 of $f_2(x)$. By continuity of f_1 and f_2, both $f_1^{-1}(V_1)$ and $f_2^{-1}(V_2)$ are neighborhoods of x, and so is $U = f_1^{-1}(V_1) \cap f_2^{-1}(V_2)$. Since $x \in \bar{A} \setminus A$, the neighborhood U contains some a in A such that $f_1(a) = f(a) = f_2(a)$. Therefore, $f(a) \in V_1 \cap V_2$ which is assumed empty.

(4) **Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $X = \prod_{\alpha \in J} X_\alpha$.**

(a) **Give a necessary and sufficient condition for a sequence $\{u_n\}_{n \in \mathbb{Z}_+}$ to converge in X equipped with the product topology.**

Assume that $\lim_{n\to\infty} u_n = l$. The projection maps π_α are continuous so the 'non-necessarily metrizable' part of the sequential characterization theorem [M. Th. 21.3] implies that

$$(\star) \qquad \forall \alpha \in J \quad , \quad \lim_{n\to\infty} u_{n\alpha} = l_\alpha .$$

Conversely, assume that $\lim_{n\to\infty} \pi_\alpha(u_n) = \pi_\alpha(l)$ for every $\alpha \in J$ and let U be a neighborhood of l. We may assume that U is an intersection of cylinders, that is,

$$U = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \ldots \cap \pi_{\alpha_p}^{-1}(U_{\alpha_p})$$

since such elements form a basis for the product topology. With our assumption, there exists, for each $i \in \{1, \ldots, p\}$, a rank N_i such that $\pi_{\alpha_i}(u_n) \in U_{\alpha_i}$ for all $n \geq N_i$. This implies that $u_n \in U$ for all $n \geq \max_{1 \leq i \leq n} N_i$, so $\lim_{n\to\infty} u_n = l$.

(b) **Does the result hold if X is equipped with the box topology?**

The box topology is finer than the product topology so condition $(\star)$ is certainly necessary. It is not sufficient, however, as the following example shows. Let $J = \mathbb{Z}_+$ and $X_k = \mathbb{R}$ with the standard topology for each $k \in \mathbb{Z}_+$ so that $X = \mathbb{R}^\omega$ as a set. Then, consider the sequence $({}^n u)_{n \in \mathbb{Z}_+}$ defined by

$${}^n u_k = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases} .$$

Then $\lim_{n\to\infty} {}^n u_k = \lim_{n\to\infty} \pi_k({}^n u) = 0$ for every $k \in \mathbb{Z}_+$ so ${}^n u$ converges to the zero sequence in the product topology.
On the other hand, the open box $\prod_{k \in \mathbb{Z}_+} (-1, 1)$ is a neighborhood of the zero sequence that contains no term of the sequence $({}^n u)_{n \in \mathbb{Z}_+}$, which therefore cannot converge to zero in the box topology.