Math 54 Summer 2015
Homework #5: continuous maps, the product topology

(1) (a) Consider $\mathbb{Z}_+$ equipped with the topology in which open sets are the subsets U such that if n is in U, then any divisor of n belongs to U. Give a necessary and sufficient condition for a function $f : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ to be continuous.

(b) Let $\chi_{\mathbb{Q}}$ be the indicator of $\mathbb{Q}$. Prove that the map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\varphi(x) = x \cdot \chi_{\mathbb{Q}}(x)$ is continuous at exactly one point.

(2) Let X and Y be topological spaces. If A is a subset of either, we denote by A' the sets of accumulation points of A and by ∂A its boundary.
Let $f : X \longrightarrow Y$ be a map. Determine the implications between the following statements:

(i) f is continuous.

(ii) $f(A') \subset (f(A))'$ for any $A \subset X$.

(iii) $\partial(f^{-1}(B)) \subset f^{-1}(\partial B)$ for any $B \subset Y$.

(3) Let X and Y be topological spaces, and assume Y Hausdorff. Let A be a subset of X and f_1, f_2 continuous maps from the closure $\bar{A}$ to Y.
Prove that if f_1 and f_2 restrict to the same function $f : A \to Y$, then $f_1 = f_2$.

(4) Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces and $X = \prod_{\alpha \in J} X_\alpha$.

(a) Give a necessary and sufficient condition for a sequence $\{u_n\}_{n \in \mathbb{Z}_+}$ to converge in X equipped with the product topology.

(b) Does the result hold if X is equipped with the box topology?