Math 54 Summer 2015
Homework #4: closed sets and limit points - Elements of solution

(1) **Prove the following result:**
Theorem Let X be a set and $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ a map such that
(i) $\gamma(\emptyset) = \emptyset$;
(ii) $A \subset \gamma(A)$;
(iii) $\gamma(\gamma(A)) = \gamma(A)$;
(iv) $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$.
Then the family $\mathcal{T} = \{X \setminus \gamma(A)\,,\, A \subset X\}$ is a topology in which $\overline{A} = \gamma(A)$.

First, we prove that $A \subset B \Rightarrow \gamma(A) \subset \gamma(B)$. To do so, observe that $A \subset B$ is equivalent to $A \cup B = B$. Therefore, $\gamma(B) = \gamma(A \cup B) \overset{\textbf{(iv)}}{=} \gamma(A) \cup \gamma(B) \supset \gamma(A)$.

(O1) The subset $X = X \setminus \emptyset \overset{\textbf{(i)}}{=} X \setminus \gamma(\emptyset)$ is in $\mathcal{T}$. Moreover, **(ii)** implies that $X = \gamma(X)$ so $\emptyset = X \setminus \gamma(X)$ is also in $\mathcal{T}$.

(O2) Let $\{U_\alpha\}_{\alpha \in J}$ be a family such that $U_\alpha = X \setminus \gamma(A_\alpha)$ for each $\alpha \in J$, and $U = \bigcup_{\alpha \in J} A_\alpha$. De Morgan's Laws imply that

$$X \setminus U = \bigcap_{\alpha \in J} A_\alpha$$

and we want to prove that this set is of the form $\gamma(B)$ for some subset B of X. Since $\bigcap_{\alpha \in J} \gamma(A_\alpha) \subset \gamma(A_\alpha)$ for all $\alpha \in J$, and γ preserves inclusions, we get, for all $\alpha \in J$,

$$\gamma(X \setminus U) \subset \gamma(\gamma(A_\alpha)) \overset{\textbf{(iii)}}{=} \gamma(A_\alpha)$$

so that $\gamma(X \setminus U) \subset \bigcap_{\alpha \in J} \gamma(A_\alpha) = X \setminus U$, the reverse inclusion is guaranteed by **(ii)**, hence $X \setminus U = \gamma(X \setminus U)$, that is,

$$U = X \setminus \gamma(X \setminus U)$$

and $\mathcal{T}$ is stable under arbitrary unions.

(O3) Let $\{U_i = X \setminus \gamma(A_i)\}_{1 \le i \le n}$ be a finite family of elements of $\mathcal{T}$. De Morgan's Laws imply that

$$X \setminus \bigcap_{i=1}^{n} U_i = X \setminus \bigcap_{i=1}^{n} \gamma(A_i) = X \setminus \gamma\left(\bigcap_{i=1}^{n} A_i\right)$$

where the last equality follows from **(iv)** by induction. This shows that $\mathcal{T}$ is stable under finite intersections, which concludes the proof that it is a topology on X.

Let A be a subset of X. Then $\gamma(A)$ is closed by definition of $\mathcal{T}$ and $A \subset \gamma(A)$ by **(ii)** so $\bar{A} \subset \gamma(A)$. Conversely, observe that $X \setminus \bar{A}$, being open, is of the form $X \setminus \gamma(B)$, that is, $\bar{A} = \gamma(B)$ for some $B \subset X$. Since $A \subset \bar{A}$, and γ preserves inclusions, it follows that

$$\gamma(A) \subset \gamma(\bar{A}) = \gamma(\gamma(B)) \overset{\textbf{(iii)}}{=} \gamma(B) = \bar{A},$$

hence $\gamma(A) = \bar{A}$.

(2) (a) **Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x)\ ,\ x \in X\}$ is closed in $X \times X$.**

A key observation is that for A and B subsets of X, the condition $A \cap B = \emptyset$ is equivalent to $(A \times B) \cap \Delta = \emptyset$.
Now, assume X Hausdorff and let $(x,y) \in (X \times X) \setminus \Delta$. Since $x \neq y$, there exist disjoint open sets $U_x \ni x$ and $U_y \ni y$. By definition of the product topology, $U = U_x \times U_y$ is a neighborhood of (x,y) and by the preliminary observation, $U \cap \Delta = \emptyset$ so $X \times X \setminus \Delta$ is open hence Δ is closed.
Conversely, assume that Δ is closed and let $x \neq y$ in X. Since (x,y) belongs to $(X \times X) \setminus \Delta$ assumed open, there exists a neighborhood V of (x,y) such that $V \cap \Delta = \emptyset$. Product of open sets form a basis for the topology of $X \times X$, so there exist open sets U_1 and U_2 such that $(x,y) \in U_1 \times U_2 \subset V$ so $(U_1 \times U_2) \cap \Delta = \emptyset$ which, by the preliminary observation again, guarantees that U_1 and U_2 are disjoint neighborhoods of x and y respectively.

(b) **Determine the accumulation points of $A = \left\{\frac{1}{m} + \frac{1}{n}\ ,\ m,n \in \mathbb{Z}_+\right\} \subset \mathbb{R}$.**

Let A' denote the set of accumulation points of A. The fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ implies that $\left\{\frac{1}{p}\ ,\ p \in \mathbb{Z}_+\right\} \cup \{0\} \subset A'$. Let us prove the converse inclusion.
First, observe that if an interval (a,b) with $a > 0$ contains infinitely many elements of the form $\frac{1}{m} + \frac{1}{n}$, then one of the variables m and n must take only finitely many values, while the other takes infinitely many values. Now let $x \in A'$ with $x > 0$. For any $\varepsilon > 0$, the set $B_\varepsilon = (x - \varepsilon, x + \varepsilon) \cap A$ must be infinite. Without loss of generality, we can assume that

$$B_\varepsilon = \left\{\frac{1}{m} + \frac{1}{n}\ ,\ m \in F\ ,\ n \in I_m\right\}$$

with F finite and at least one I_m infinite, say I_{m_0}. For all $n \in I_{m_0}$, we have

$$\left|\left|x - \frac{1}{m_0}\right| - \frac{1}{n}\right| \leq \left|x - \frac{1}{m_0} - \frac{1}{n}\right| < \varepsilon.$$

For n large enough, the left-hand side can be made arbitrarily close to $\left|x - \frac{1}{m_0}\right|$, in particular, we get that $\frac{1}{2}\left|x - \frac{1}{m_0}\right| < \varepsilon$. If $x > 0$ is not of the form $\frac{1}{m_0}$ for any $m_0 \in \mathbb{Z}_+$, then there exists a positive minimum value for the numbers $\frac{1}{2}\left|x - \frac{1}{m_0}\right|$ and B_ε cannot be infinite for arbitrarily small values of ε.

(3) **The boundary of a subset A in a topological space X is defined by**

$$\partial A = \bar{A} \cap \overline{X \setminus A}.$$

(a) **Show that $\bar{A} = \mathring{A} \sqcup \partial A$**[1]**.**

If $x \in \mathring{A}$, there exists a neighborhood of A that is included in A. If $x \in \partial A$, in particular $x \in \overline{X \setminus A}$ so every neighborhood of x intersects $X \setminus A$. This is a contradiction so $\mathring{A} \cap \partial A = \emptyset$.

The interior and boundary of A are included in $\bar{A}$ by definition so the inclusion $\bar{A} \supset \mathring{A} \sqcup \partial A$ is trivial. Conversely, let $x \in \bar{A}$. If x has a neighborhood U such that $U \subset A$, then $x \in \mathring{A}$. The alternative is that every neighborhood of x has non-empty intersection with $X \setminus A$, that is $x \in \overline{X \setminus A}$ so that $x \in \partial A$. Therefore, $\bar{A} \subset \mathring{A} \sqcup \partial A$, which concludes the proof.

(b) **Show that $\partial A = \emptyset$ if and only if A is open and closed.**

By definition of the interior and the closure, $\mathring{A} \subseteq A \subseteq \bar{A}$ and A is open and closed if and only if $\mathring{A} = \bar{A}$. By (a), this is equivalent to $\partial A = \emptyset$.

(c) **Show that U is open if and only if $\partial U = \bar{U} \setminus U$.**

The result of (a) states that U and $\mathring{U}$ are complements in $\bar{U}$, so $\mathring{U} = \bar{U} \setminus \partial U$ and U is equal to $\mathring{U}$, that is, U is open if and only if $U = \bar{U} \setminus \partial U$, which is equivalent to the condition $\partial U = \bar{U} \setminus U$.

(d) **If U is open, is it true that $U = \mathring{\bar{U}}$?**

If U is open, the inclusion $U \subset \bar{U}$ implies that $U \subset \mathring{\bar{U}}$. However, the reverse inclusion may fail: consider for instance $U = \mathbb{R} \setminus \{0\}$ in $\mathbb{R}$. It is open as the union of open intervals and $\bar{U} = \mathbb{R}$ so that $\mathring{\bar{U}} = \mathbb{R} \supsetneq U$.

[1] The *disjoint union* symbol $\sqcup$ is used to indicate that the sets in the union have empty intersection.

(4) **Find the boundary and interior of each of the following subsets of $\mathbb{R}^2$.**

(a) $A = \{(x, y)\ ,\ y = 0\}$
(b) $B = \{(x, y)\ ,\ x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \mathbb{Q} \times \mathbb{R}$
(e) $E = \{(x, y)\ ,\ 0 < x^2 - y^2 \leq 1\}$
(f) $F = \{(x, y)\ ,\ x \neq 0 \text{ and } y \leq \frac{1}{x}\}$

Note that, except for (d), a picture is very helpful to determine the boundary and interior of the subsets at hand before rigorously justifying the intuition, using what is known about the (metric) topology of $\mathbb{R}^2$.

(a) Observe that A is closed, as the complement of $\mathbb{R} \times (-\infty, 0) \cup (0, +\infty)$ which is open as a product of open sets. Another way to see this is to remark that every element of $\mathbb{R}^2 \setminus A$ is of the form (x, y) with $y \neq 0$, and for any $x \in \mathbb{R}$, the basis element

$$V = (x - 1, x + 1) \times \left(y - \frac{|y|}{2}, y + \frac{|y|}{2}\right)$$

satisfies $(x, y) \in V \subset \mathbb{R}^2 \setminus A$.
Moreover, the interior of A is empty: every element of A is of the form $(x, 0)$, any neighbourhood of which contains a basis element $(a, b) \times (c, d)$ with $c < 0 < d$, which in turn cannot be included in A, for it contains $(x, \frac{d}{2}) \notin A$.
We conclude that $\mathring{A} = \emptyset$ and $\partial A = A$.

(b) Note that $B = (0, +\infty) \times (-\infty, 0) \cup (0, +\infty)$ is open as a product of open sets. Another way to see this is to consider $(x, y) \in B$, that is, $x > 0$ and $y \neq 0$. Then

$$V = \left(\frac{x}{2}, \frac{3x}{2}\right) \times \left(y - \frac{|y|}{2}, y + \frac{|y|}{2}\right)$$

is a neighborhood of (x, y) that is contained in B, which is therefore open.
Finally, B is open because it is the inverse image of $\mathbb{R}^2 \setminus A$ open under the continuous map $(x, y) \mapsto (\ln x, y)$.
Let us prove that the closure of B is the closed half-plane R defined by $x \geq 0$. Let V be a neighborhood of $(x, y) \in R$. If $(x, y) \in B$, there is nothing to prove. If $xy = 0$, then V contains a subset of the form $(a, b) \times (c, d)$ with $0 < b$ and $cd \neq 0$ so $\left(\frac{x+b}{2}, \frac{y+d}{2}\right)$ or $\left(\frac{x+b}{2}, \frac{y+c}{2}\right)$ belongs to $V \cap B$, which is therefore not empty. We have proved that $R \subset \bar{B}$. The converse inclusion follows from the same argument invoked to prove that $\mathbb{R}^2 \setminus A$ is open.
Since B is open, it follows from (c) in the previous problem that $\partial B = \bar{B} \setminus B$, that is ∂B is the union of the vertical axis and the positive horizontal axis.

(c) Since $\overline{A \cup B} = \bar{A} \cup \bar{B}$, it follows form (a) and (b) that $\bar{C} = R \cup A$ consists of the points (x, y) such that $x \geq 0$ or $y = 0$.

Next, $\mathring{C}$ is the right half-plane $(0, +\infty) \times \mathbb{R}$: this set is open as the product of open sets and it is maximal. Indeed, if $x \leq 0$, then any neighborhood of (x, y) contains a subset of the form $(a, b) \times (c, d)$ with $a < 0$ and $cd \neq 0$ so $\left(\frac{x+a}{2}, \frac{y+d}{2}\right)$ or $\left(\frac{x+a}{2}, \frac{y+c}{2}\right)$ belongs to $V \cap (\mathbb{R}^2 \setminus C)$, which is therefore not empty.
It follows from the result proved in (a) of the previous problem that $\partial C = \bar{C} \setminus \mathring{C}$ is the union of the vertical axis and the negative horizontal axis.

(d) Every non-empty open interval of $\mathbb{R}$ contains infinitely many rational and irrational numbers, so every product of intervals contains infinitely many elements of D and $\mathbb{R}^2 \setminus D$. Therefore, $\partial D = \mathbb{R}^2$ and, since $\partial D = \bar{D} \setminus \mathring{D}$, it follows immediately that $\mathring{D} = \emptyset$.

(e) First, observe that the set $\Omega = \{(x, y)\ ,\ 0 < x^2 - y^2 < 1\}$ is open, for instance as the inverse image of the open set $(0, 1)$ under the map $(x, y) \mapsto x^2 - y^2$, which is polynomial, hence continuous.
A similar argument, shows that $\Gamma = \{(x, y)\ ,\ 0 \leq x^2 - y^2 \leq 1\}$ is closed. Since $\Omega \subset E \subset \Gamma$, we get the chain of inclusions $\Omega \subset \mathring{E} \subset \bar{E} \subset \Gamma$, hence

$$\partial E = \bar{E} \setminus \mathring{E} \subset \Gamma \setminus \Omega.$$

In other words, a boundary point (x, y) of E satisfies either $x^2 = y^2$ or $x^2 - y^2 = 1$. Conversely, assume that $x^2 - y^2 = 1$. Every neighbourhood of (x, y) contains the points $P_\delta = (x + \delta, y)$ for $\delta \in (-\delta_0, \delta_0)$ with $\delta_0 > 0$. Since

$$(x + \delta)^2 - y^2 = 1 + 2\delta(x + \delta),$$

and the quantity $2\delta(x + \delta)$ takes arbitrarily small positive values when δ runs over $(-\delta_0, \delta_0)$, we see that there are points P_δ in $\mathbb{R}^2 \setminus E$ and E so (x, y) is a boundary point of E. One can proceed in the same way to verify that the two lines given by the equation $x^2 = y^2$ are also included in ∂E, which concludes the proof that ∂E consists exactly of the union of the hyperbola with equation $x^2 - y^2 = 1$ and the lines with equations $y = \pm x$.
It also follows that $\mathring{E} = \Omega$. We have already obtained the inclusion $\Omega \subset \mathring{E}$. Conversely, assume that (x, y) is a point in E not in Ω. Then $x^2 - y^2 = 1$ so (x, y) belongs to ∂E which is disjoint from $\mathring{E}$. This proves that $\mathring{E} \subset \Omega$ and the equality.

(f) No new technique is needed to prove that $\mathring{F}$ is the region located strictly below the branches of the hyperbola with equation $xy = 1$, that is,

$$\mathring{F} = \left\{(x, y)\ ,\ x \neq 0 \text{ and } y < \frac{1}{x}\right\},$$

and that ∂F is the union of the hyperbola and the vertical axis:

$$\mathring{F} = \left\{(x, y)\ ,\ x = 0 \text{ or } y = \frac{1}{x}\right\}.$$