Math 54 Summer 2015
Homework #2: metric spaces - Elements of solution

(1) **Balls.**

a. Consider $\mathbb{Z} \times \mathbb{Z}$ equipped with the Euclidean metric. Describe $\mathcal{B}\left((3,2), \sqrt{2}\right)$ and $\mathcal{B}_c\left((3,2), \sqrt{2}\right)$.

One can enumerate the elements:

$$\mathcal{B}\left((3,2), \sqrt{2}\right) = \{(2,2); (3,1); (3,2); (3,3); (4,2)\}.$$

and

$$\mathcal{B}_c\left((3,2), \sqrt{2}\right) = \mathcal{B}\left((3,2), \sqrt{2}\right) \cup \{(2,1); (2,3); (4,1); (4,3)\}$$

b. Let X be a set equipped with the discrete metric and x in X. Describe the balls $\mathcal{B}(x,r)$ for all $r > 0$.

By definition, $\mathcal{B}(x,r) = \{x\}$ for $0 < r \leq 1$ and $\mathcal{B}(x,r) = X$ for $r > 1$.

(2) **Continuous maps.**

a. Prove that the map f defined on $\mathbb{R}$ by $f(x) = x^2 + 1$ is continuous.

Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Note that if $a - 1 \leq x \leq a + 1$, then $|x + a| \leq 2|a| + 1$. Therefore, since $|f(x) - f(a)| = |x - a||x + a|$, we get, for $x \in [a-1, a+1]$,

$$|f(x) - f(a)| \leq |x - a|(2|a| + 1)$$

and it suffices to choose $|x - a| < \min\{\frac{\varepsilon}{2|a|+1}, 1\}$ to guarantee $|f(x) - f(a)| < \varepsilon$.

b. Let E_1, E_2, E_3 be metric spaces and $u : E_2 \to E_3$, $v : E_1 \to E_2$ be continuous maps. Prove that $u \circ v$ is continuous.

Let Ω be open in E_3 and apply Theorem [MC] twice: $u^{-1}(\Omega)$ is open in E_2 by continuity of u and $(u \circ v)^{-1}(\Omega) = v^{-1}(u^{-1}(\Omega))$ is open by continuity of v.

(3) **Let (E,d) be a metric space. Prove that a subset $\Omega \subset E$ is open if and only if for every point $x \in \Omega$, there exists an open ball containing x and included in Ω.**

The definition seen in class for open sets in a metric space differs only by the fact that it requires the ball to be centered at the point considered. Therefore, open sets trivially satisfy the property.
Observe that if a point x is included in a ball $\mathcal{B}(a,r)$, the triangle inequality implies that $\mathcal{B}(x, r - d(a,x))$ is included in $\mathcal{B}(a,r)$. The converse follows.

(4) **Let (E, d) be a metric space and $A \subset E$. A point a in A is called *interior* if there exists $r > 0$ such that any point x in E such that $d(a, x) < r$ is in A. The set $\overset{o}{A}$ of interior points of A is called the *interior of* A.**

a. Prove that $\overset{o}{A}$ is the union of all the open balls contained in A.

Let $\dot{A}$ be the union of all the open balls contained in A and let a be in $\dot{A}$. By definition of $\dot{A}$ and the argument used in (3), there exists a ball $\mathcal{B}(a, r)$ included in A, so $\dot{A} \subset \overset{o}{A}$. Conversely, let a be in $\overset{o}{A}$. By definition, there exists $r > 0$ such that $\mathcal{B}(a, r) \subset A$ so $a \in \dot{A}$, hence the result.

b. Prove that $\overset{o}{A}$ is the largest open subset contained in A.

First, $\overset{o}{A}$ is open as the union of open subsets, as proved in **a.** We argue by contradiction: assume the existence of Ω open such that $\overset{o}{A} \subsetneq \Omega \subset A$. Let $x \in \Omega \setminus \overset{o}{A}$. Since $x \notin \overset{o}{A}$, no ball $\mathcal{B}(x, r)$ with $r > 0$ is contained in A. Since Ω is open, it must contain such a ball, which contradicts the assumption that $\Omega \subset A$.

c. Can $\overset{o}{A}$ be empty if A is not?

Yes: consider for instance $A = \mathbb{Z}$ in $E = \mathbb{R}$, or any strict linear subspace of $\mathbb{R}^n$ and observe that every ball centered at the origin must contain a basis.