Math 54 Summer 2015
Homework #2: metric spaces

(1) **Balls.** *No proof is required for this problem.*
  a. Consider $\mathbb{Z} \times \mathbb{Z}$ equipped with the Euclidean metric.
  Describe $\mathcal{B}\left((3,2), \sqrt{2}\right)$ and $\mathcal{B}_c\left((3,2), \sqrt{2}\right)$.

  b. Let X be a set equipped with the discrete metric and x a point in X.
  Describe the balls $\mathcal{B}(x, r)$ for all $r > 0$.

(2) **Continuous maps.**
  a. Prove that the map f defined on $\mathbb{R}$ by $f(x) = x^2 + 1$ is continuous.

  b. Let (E_1, d_1), (E_2, d_2), (E_3, d_3) be metric spaces and $u : E_2 \to E_3$, $v : E_1 \to E_2$ be continuous maps. Prove that $u \circ v$ is continuous.

(3) Let (E, d) be a metric space. Prove that a subset $\Omega \subset E$ is open if and only if for every point $x \in \Omega$, there exists an open ball containing x and included in Ω.

(4) Let (E, d) be a metric space and $A \subset E$ a subset. A point a in A is called *interior* if there exists $r > 0$ such that any point x in E such that $d(a, x) < r$ is in A.
The set of interior points of A is called the *interior of* A and denoted by $\overset{\circ}{A}$.

  a. Prove that $\overset{\circ}{A}$ is the union of all the open balls contained in A.

  b. Prove that $\overset{\circ}{A}$ is the largest open subset contained in A.

  c. Can $\overset{\circ}{A}$ be empty if A is not?