MATH 54 - TOPOLOGY
SUMMER 2015
FINAL EXAMINATION

DURATION: 3 HOURS

This exam consists of 6 independent problems. Your may treat them in the order of your choosing, starting each problem on a new page.

Problem 1

1. Let X be a Hausdorff space and K_1, K_2 disjoint compact subsets of X.
Prove that there exist disjoint open sets U_1 and U_2 such that $K_1 \subset U_1$ and $K_2 \subset U_2$.

2. Let X be a discrete topological space. Describe the compact subsets of X.

Problem 2

A topological space is said *totally disconnected* if its only connected subspaces are singletons.

1. Prove that a discrete space is totally disconnected.

2. Does the converse hold?

Problem 3

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces; let $A_\alpha \subset X_\alpha$ for each $\alpha \in J$.

1. In $\prod_{\alpha \in J} X_\alpha$ equipped with the product topology, prove that

$$\prod_{\alpha \in J} \bar{A}_\alpha = \overline{\prod_{\alpha \in J} A_\alpha}.$$

2. Does the result hold if $\prod_{\alpha \in J} X_\alpha$ carries the box topology?

Problem 4

Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?

Problem 5

Let (E,d) be a metric space. An *isometry of* E is a map $f : E \longrightarrow E$ such that

$$d\left(f(x), f(y)\right) = d(x,y)$$

for all $x, y \in E$.

1. Prove that any isometry is continuous and injective.

Assume from now on that E is compact and f an isometry. We want to prove that f is surjective. Assume to the contrary the existence of $a \notin f(E)$.

2. Prove that there exists $\varepsilon > 0$ such that $B(a,\varepsilon) \subset E \setminus f(E)$.

3. Consider the sequence defined by $x_1 = a$ and $x_{n+1} = f(x_n)$. Prove that

$$d(x_n, x_m) \geq \varepsilon$$

for $n \neq m$ and derive a contradiction.

4. Prove that an isometry of a compact metric space is a homeomorphism.

Problem 6

Let X be a set, $\mathscr{P}(X)$ the set of subsets of X and $\iota : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ a map satisfying:

$$(1) \qquad \iota(X) = X$$

$$(2) \qquad \iota(A) \subset A$$

$$(3) \qquad \iota \circ \iota(A) = \iota(A)$$

$$(4) \qquad \iota(A \cap B) = \iota(A) \cap \iota(B)$$

for all $A, B \subset X$.

1. Check that $A \subset B \Rightarrow \iota(A) \subset \iota(B)$ for $A, B \subset X$.

2. Prove that the family $\mathcal{T} = \{\iota(A)\ ,\ A \in \mathscr{P}(X)\}$ is a topology on X.

3. Prove that, in this topology, $\mathring{A} = \iota(A)$ for all $A \subset X$.