MATH 54 SUMMER 2015

DIARY

Textbook

[**M**] *Topology (2^nd ed.)*, by J. Munkres.

Effective syllabus

O. Introduction - *4 lectures*

O.1 Elementary set theory
O.2 Basics on metric spaces

I. Topological spaces and continuous functions - *16 lectures*

I.1. Topologies
I.2. Bases and subbases
I.3. Closed sets
I.4. Continuous maps and the category **Top**.
I.5. Topologies on cartesian products
I.6. Metrizable topologies

II. Connectedness - *2 lectures*

II.1. Connected spaces
II.2. Path connectedness
II.3. Connected components

III. Compactness - *5 lectures*

III.1. Compact spaces
III.2. Fréchet and sequential compactness
III.3. Local compactness, Alexandrov Compactification

IV. Other topics - *3 lectures*

IV.1. Separation axioms
IV.2. The Urysohn Lemma and the Urysohn Metrization Theorem
IV.3. Normed linear spaces
IV.4. Topological properties of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$

Updated: August 26, 2015.

Week 1

Lecture 1. General introduction. (Why) should we study Topology? What is it?
Poincaré (1895): *Analysis situs*.

> There is a science called *analysis situs* and **which has for its object the study of the positional relations of the different elements of a figure, apart from their sizes. This geometry is purely qualitative; its theorems would remain true if the figures, instead of being exact, were roughly imitated by a child**. [...] The importance of *analysis situs* is enormous and can not be too much emphasized [...].
>
> H. Poincaré, *The value of Science*, 1905.

Intuitive notions of neighborhood and deformation.
Fun: example of topological problem: the seven bridges of Königsberg.

Week 2

Lecture 2. [**M**, §6-7]
Finite sets and cardinality. Examples and comparison of infinite subsets of $\mathbb{Z}_+$ (evens, odds, squares, primes). Bijection between two line segments.
Countably infinite and countable sets. Examples: $\mathbb{Z}$ and $\mathbb{Z}_+ \times \mathbb{Z}_+$ are countably infinite.
Fun: the set $S = \{n^2\ ,\ n \in \mathbb{Z}_+\}$ is in bijection with $\mathbb{Z}$. However, there are 'more' integers than squares in the sense that $\sum\limits_{k\in\mathbb{Z}_+} \frac{1}{k} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges while $\sum\limits_{k\in S} \frac{1}{k} = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.
What about $\sum\limits_{p\,\text{prime}} \frac{1}{p}$?

Lecture 3. [**M**, §7]
Characterization of countable sets: B is countable iff $\mathbb{Z}_+ \xrightarrow{\text{surj.}} B$ iff $B \xrightarrow{\text{inj.}} \mathbb{Z}_+$.
Infinite subset of $\mathbb{Z}_+$ are countable. Examples of countable sets:

- $\mathbb{Z}_+ \times \mathbb{Z}_+$, $\mathbb{Q}$,...
- subsets, *countable* unions, *finite* products of countable sets.

There exist uncountable sets: $\{0,1\}^\omega$ (diagonal extraction argument) and $\mathbb{R}$ for instance.

Week 3

Lecture 4. Notion of distance, metrics. Examples of metric spaces: $(\mathbb{R}^2, \text{Euclidean})$, $(\mathbb{R}^2, \text{Manhattan})$, $\mathbb{Z}$ with the ordinary metric, $(\mathbb{Z}, 2\text{-adic})$. Balls in metric spaces.
Continuous functions on $\mathbb{R}$: definition and expression in terms of balls and inverse images.
Fun: p-adic metric on $\mathbb{Q}$.

Lecture 5. Open sets in metric spaces. Examples in $(\mathbb{R}, \text{Euclidean})$.
Theorem [MT]: let (E,d) be a metric space. Then,

- E and $\emptyset$ are open;
- arbitrary unions of open sets are open;
- finite intersections of open sets are open.

Continuous maps between metric spaces: definition in terms of balls.
Theorem [MC]: a map between metric spaces is continuous if and only if the inverse image of any open set is an open set.

Lecture 6. [**M**, §12-13]
Topology on a set. Discrete and trivial topologies. Finite complement topology.
Reformulation of Theorem [MT]: open sets defined by a metric constitute a topology.
Comparison between topologies: notion of finer (stronger) and coarser (weaker) topology.
Basis for a topology. Examples: disks and rectangles in $\mathbb{R}^2$.

Lecture 7. [**M**, §13]
Topology $\mathcal{T}(\mathcal{B})$ generated by a basis $\mathcal{B}$. Basis elements are open ($\mathcal{B} \subset \mathcal{T}(\mathcal{B})$).
Description of $\mathcal{T}(\mathcal{B})$ (Lemma 13.1): the open sets of $\mathcal{T}(\mathcal{B})$ are the unions of elements of $\mathcal{B}$.
Criterion to find a basis of a given topology $\mathcal{T}$ on a set X (Lemma 13.2):
if a subset $\mathcal{C}$ of $\mathcal{T}$ is a finer covering[1] of X, then $\mathcal{C}$ is a basis and generates $\mathcal{T}$, i.e. $\mathcal{T}(\mathcal{C}) = \mathcal{T}$.
Topologies can be compared by comparing bases (Lemma 13.1):
$\mathcal{T}(\mathcal{B}')$ is finer than $\mathcal{T}(\mathcal{B})$ if and only if $\mathcal{B}'$ is a finer covering of X than $\mathcal{B}$.

Week 4

Lecture 8. [**M**, §13-14]
Topology generated by a subbasis.
Order topology: definition, examples: $\mathbb{R}^2$ and $\{1, 2\} \times \mathbb{Z}_+$ with the lexicographical order.
Comparison with the Euclidean and the discrete topology respectively.

Lecture 9. [**M**, §15]
The product topology: definition, bases. Projections and cylinders.

Lecture 10. [**M**, §16]
The subspace topology: definition, bases. Restriction commutes to products.
The topology of the restricted order may differ from the restricted order topology: cases of $X = \mathbb{R}$ and $Y_1 = [0, 1]$, $Y_2 = [0, 1) \cup \{2\}$. They coincide in the case of a convex subset.

Lecture 11. [**M**, §17]
Closed sets: definition, examples. Properties: stability under arbitrary intersections and finite unions. A topology can be defined by its closed sets. Closed sets in the subspace topology. Closure and interior of a set, closure in the subspace topology.
A topology can be defined by its closure operation:
Theorem [Closure] Let X be a set and $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ a map such that,

- $\gamma(\emptyset) = \emptyset$;
- $A \subset \gamma(A)$;
- $\gamma(\gamma(A)) = \gamma(A)$;
- $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$.

Then the family $\{X \setminus \gamma(A)\,,\, A \in \mathcal{P}(X)\}$ is a topology in which $\overline{A} = \gamma(A)$.

[1] If $\mathcal{E}$ and $\mathcal{F}$ are families of subsets of X, we say that $\mathcal{F}$ is a *finer covering of X than $\mathcal{E}$* if for every '$x \in E \in \mathcal{E}$' there exists $F \in \mathcal{F}$ such that $x \in F \subset E$

Week 5

Lecture 12. [**M**, §17]
Characterization of the closure and accumulation points.

Lecture 13. [**M**, §17]
Theorem [MUL] In a metric space, if a sequence converges, it has a unique limit.
Convergent sequences in topological spaces. Uniqueness of limits in Hausdorff (T_2) spaces.
Characterization of accumulation points in T_1 spaces.

Midterm 1. Divisor topology on $\mathbb{Z}_+$. Equivalent metrics generate the same topology. Topology generated by the union of two topologies. Examples of subspaces of $\mathbb{R}$, $[-1,1]$ and $\mathbb{R}_\ell \times \mathbb{R}_u$.
Sperner's Lemma and consequences, by J. Bass.

Guest lecture. *The problem with topology*, by James Binkoski (Dartmouth College), an introduction to T. Maudlin's *New foundations for physical geometry*.

Week 6

Lecture 14. [**M**, §18]
Continuous maps between general topological spaces. Any function $f : X \to Y$ can be made continuous by equipping X with the discrete topology or Y with the trivial topology.
Characterization at the level of a (sub)basis for the topology on the target space.
Characterization in terms of the direct image of the closure, inverse image of closed sets, in terms of neighborhoods.

Lecture 15. *(D. Freund)* [**M**, §18]
Construction of continuous functions, the Pasting Lemma.

Lecture 16. Homeomorphisms: definition, examples, topological properties.
Categories, definitions and examples: **Set**, **Mod**($\mathbb{R}$), **Top**, ordered sets: $\mathbb{N}$ and $\mathbf{Op}_X$.
Isomorphisms in a category. The isomorphisms in **Top** are the homeomorphisms.

Lecture 17. [**M**, §19]
Cartesian products of arbitrary indexed families of sets.
The box topology and the product topology: definition, comparison, bases.
A product of Hausdorff spaces is Hausdorff. The closure of a product is the product of the closures. Continuous maps into products.

Week 7

Lecture 18. [**M**, §20]
Metrizable topologies. Equivalent metrics generate the same topology. Example: the Euclidean and L^∞ metrics are equivalent and generate the product topology on $\mathbb{R}^n$.
Topologically equivalent metrics need not be equivalent: any metric is topologically equivalent to its associated standard bounded metric.
Generalization of the L^∞ topology: the uniform metric and topology on $\mathbb{R}^J$.

Lecture 19. [**M**, §20]
The uniform topology on $\mathbb{R}^J$ is intermediate between the product and box topologies.
It is metrizable if J is countable.

Lecture 20. [**M**, §21]
Sequential characterization of closure points and continuous maps in metric spaces.
Pointwise and uniform convergence a uniform limit of continuous functions is continuous.
Uniform convergence is equivalent to convergence in the uniform topology.
Functors between categories. Definition. Examples: $\mathbb{F}\mathrm{or} : \mathbf{Top} \longrightarrow \mathbf{Set}$, the 'matrix functor' $\mathbb{N} \longrightarrow \mathbf{Mod}^f(\mathbb{R})$.

Lecture 21. *(D. Freund)* [**M**, §23-24]
Definition: separations and connected spaces. Examples of connected spaces: $\mathbb{R}_\ell$, $\mathbb{Q}$.
Examples of disconnected spaces: $(\mathbb{Z}, \mathcal{T}_{\text{f.c.}})$, any space with the trivial topology.
Linear continua in the order topology and their intervals are connected.
A space is connected if and only if it does not have non-trivial open and closed subsets. A connected subspace lies entirely in one component of any separation of the ambient space.
The union of connected subspaces with a common point is connected.
If A is connected and $A \subseteq B \subseteq \bar{A}$, then B is connected.

Week 8

Lecture 22. [**M**, §23-25]
The continuous image of a connected space is connected. Finite products of connected spaces are connected, $\mathbb{R}^\infty$ is not connected in the box topology.
The Intermediate Value Theorem.
Path connectedness is a strictly stronger property than connectedness.
Connected components, every space is the disjoint union of its connected components.

Lecture 23. [**M**, §26]
Covers, Borel-Lebesgue definition of compact spaces. Examples: $\mathbb{R}$ and $(0,1]$ are not compact, $\{0\} \cup 1/\mathbb{Z}_+$ is.
Closed subsets of compacts are compact. Compact subspaces of Hausdorff sets are closed.

Midterm 2. Characterization of T_1 spaces. Interior and boundary of $\{(x,y) \in \mathbb{R}^2 \ , \ 0 \leq y < x^2 + 1\}$.
The metric topology is the coarsest topology making the distance continuous.
The box topology on $\mathbb{R}^\omega$ is not metrizable.
Convergence in the uniform topology is equivalent to uniform convergence.
Closure of bounded and finitely supported sequences in the uniform topology.
Furstenberg's proof of the infinitude of primes, by N. Ezroura.
Topological groups: translation are homeomorphisms, open subgroups are closed.

Lecture 24. [**M**, §26]
Compact subsets of Hausdorff spaces are compact. Points can be separated from compacts by disjoint open sets in Hausdorff spaces. Continuous images of compact sets are compact.
The Tube Lemma. Finite products of compacts are compacts.

Week 9

Lecture 25. [**M**, §27]
Segments in a totally ordered set with the least upper bound property (such as $\mathbb{R}$) are compact in the order topology. Compact sets of $\mathbb{R}^n$ in any topology associated with a metric equivalent to the ℓ^∞ metric are exactly the closed and bounded subsets.
The Extreme Value Theorem.

Lecture 26. [**M**, §28]
Féchet (limit point) compactness. Compact spaces are Fréchet compact.
Sequential compactness (Bolzano-Weierstraß property). The three notions are equivalent in metric spaces.

Lecture 27. [**M**, §29]
Locally compact Hausdorff spaces. A punctured compact Hausdorff space is locally compact and Hausdorff. Alexandrov compactification of locally compact Hausdorff spaces.

Lecture 28. [**M**, §30-34]
Generalization of sequential characterizations: nets in topological spaces.
Regular and normal spaces. First and second countability: definitions, examples and characterization in terms of closure of neighborhoods. Sequential characterizations of closure points and continuity in first countable spaces. Regular second countable spaces, metrizable spaces and compact Hausdorff spaces are normal.
The Urysohn Lemma and the Urysohn Metrization Theorem.

Week 10

Lecture 29. Topology of matrix spaces I.
Normed linear spaces over $k = \mathbb{R}$ or $\mathbb{C}$, metrics associated with norms. Equivalent norms induce equivalent metrics. In finite dimension, all norms are equivalent.
Exercise: characterizations of continuity for linear maps.
Case of $M_n(k) \subset k^{n^2}$: operator norms are submultiplicative. Continuity of operations: linear combinations, products, determinants, transposition, comatrices. The group $\mathrm{GL}(n,k)$ of invertible elements in $M_n(k)$ is a topological group. It is open in $M_n(k)$.

Lecture 30. Topology of matrix spaces II.
Density of $\mathrm{GL}(n,k)$ in $M_n(k)$ and applications: AB and BA have the same characteristic polynomial hence the same eigenvalues for any A, B in $M_n(k)$; there exists a basis of $M_n(k)$ that consists only of invertible matrices.
Connectedness properties: $\mathrm{GL}(n,\mathbb{R})$ is not connected but $\mathrm{GL}(n,\mathbb{C})$ is path connected.