Barnett 10/1/09

# MATH 53 : proof of exponential smallness of itinerary subintervals.

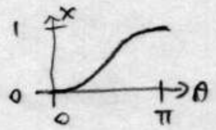
⟶ needed to prove sens. dep. in c ⟶ we'll do it again later in Ch. 3.

[This will also explain 1.15 from HW1 — the trig one!]

It turns out if you map $x$ to a new variable $\theta$, then the logistic map $G(x) = 4x(1-x)$ becomes ridiculously simple: just doubling $\theta$.

I.e:

$$
\begin{tikzcd}
x_n \arrow[r, "G"] \arrow[d, leftrightarrow, "\text{bijection}", "x=\frac{1-\cos\theta}{2}"'] & x_{n+1} = 4x_n(1-x_n) \arrow[d, leftrightarrow, "x=\frac{1-\cos\theta}{2}"] \\
\theta_n \arrow[r] & \theta_{n+1} = 2\theta_n
\end{tikzcd}
$$



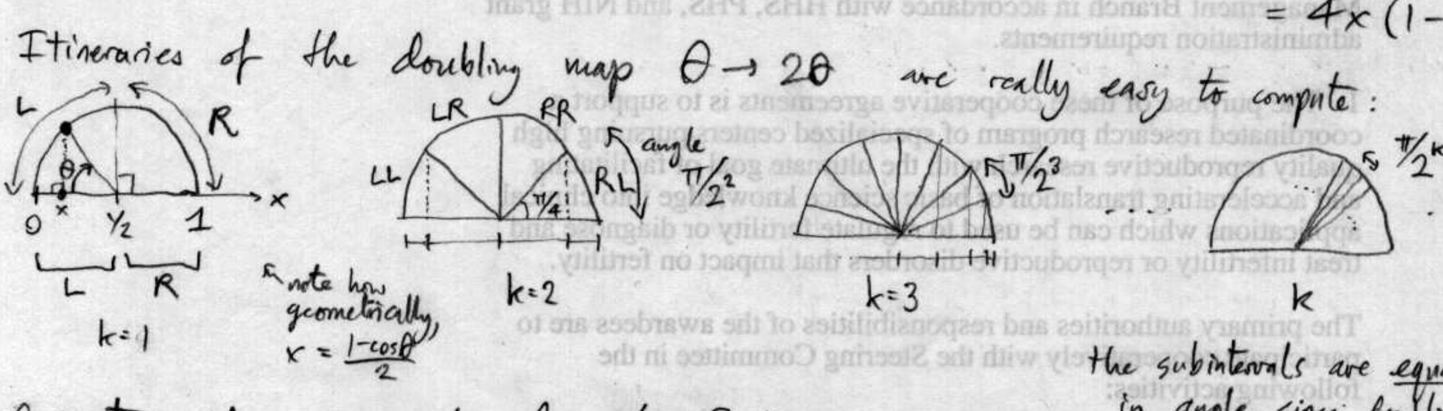

← (called a 'commutative diagram')

this states that iterating $x_n$ to $x_{n+1}$ via $G$ is equivalent to converting to $\theta$, doubling $\theta$, then converting back again. (repeat this giving $\theta \to 2^k\theta$ for HW1 1.15)

Why does this work? If $x = \frac{1-\cos\theta}{2}$ then $\underbrace{\frac{1-\cos 2\theta}{2}}_{\text{doubling } \theta} \underset{\text{by trig.}}{=} \sin^2\theta = 1-\cos^2\theta \overset{\text{by diff of squares}}{=} 4\left(\frac{1-\cos\theta}{2}\right)\left(\frac{1+\cos\theta}{2}\right)$

$= 4x(1-x)$

Itineraries of the doubling map $\theta \to 2\theta$ are really easy to compute:




the subintervals are equally spaced in angle since doubling is simple.

Projecting the semicircle down to [0,1], we see that no subinterval of $k$ symbols may exceed $\underset{\text{radius}}{\frac{1}{2}} \cdot \underset{\text{angle}}{\frac{\pi}{2^k}} = \frac{\pi}{2^{k+1}}$ in length along $x$.

QED.