MATH 53 WORKSHEET : Box-counting dimension

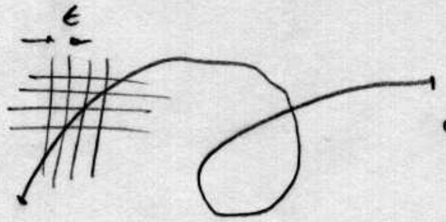
Barnett
11/2/07.
& 10/27/11

$$\boxed{\text{boxdim}(S) := \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}}$$

with 3 simplifications from lecture...

Find (and prove it you can) the boxdim for the following sets:

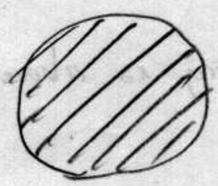
A)



curve of length L

[Hint: is there a rigorous upper bound on the number of boxes the curve can touch? Consider breaking curve into pieces each length ϵ and bounding how many boxes a piece touches.

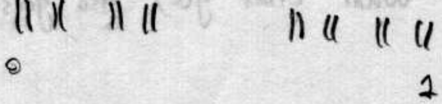
B)



a disc.

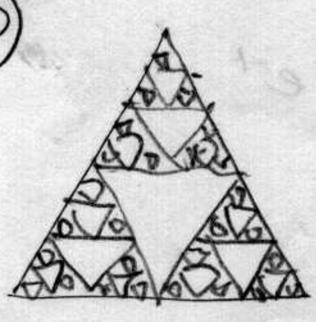
[Hint: is there a shape within which all boxes must lie?].

C)



K_∞ middle-thirds Cantor set.

D)



Sierpinski gasket

E) $K_\infty \times K_\infty \subset [0,1]^2$

MATH 53 WORKSHEET : Box-counting dimension

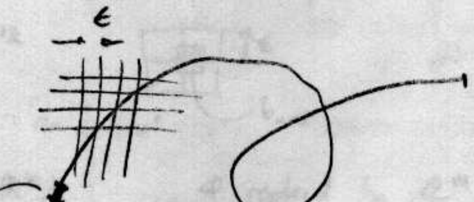
Barnett 11/2/02.

~ SOLUTIONS ~

$$\text{boxdim}(S) := \lim_{\epsilon\to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}$$

Find (and prove if you can) the boxdim for the following sets:

A)



curve of length L

[note, in order to get lower bnd on $N(\varepsilon)$ would need to prevent curve from coiling up into small # of boxes, need more info

[Hint: is there a rigorous upper bound on the number of boxes the curve can touch? Consider breaking curve into pieces each length ϵ]

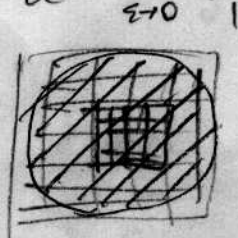
$\boxed{*}$ at most 4 boxes touched by each ε length.

$\rightarrow$ $N(\varepsilon) \le 4\frac{L}{\varepsilon}$

$$d = \lim_{\varepsilon\to 0}\frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \le \lim_{\varepsilon\to 0}\frac{\ln\frac{4L}{\varepsilon}}{\ln(1/\varepsilon)} = \frac{\ln 4L}{\cancel{\ln(1/\varepsilon)}}^{\,0} + \frac{\ln(1/\varepsilon)}{\ln(1/\varepsilon)} = 1 \quad \text{upper bnd on } d.$$

so $d = 1$.

B)



a disc.

[Hint: is there a shape within which all boxes must lie?].

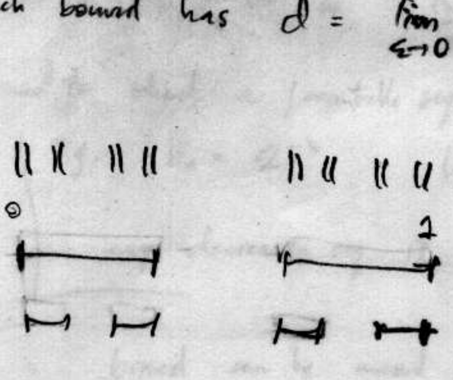
$\leftarrow$ there are squares of size L_1, L_2 such that

$$\frac{L_1^2}{\varepsilon^2} \le N(\varepsilon) \le \frac{L_2^2}{\varepsilon^2}$$

Each bound has $d = \lim_{\varepsilon\to 0}\frac{\ln(L^2/\varepsilon^2)}{\ln(1/\varepsilon)} = \lim_{\varepsilon\to 0}\frac{\ln L^2}{\cancel{\ln(1/\varepsilon)}}^{\,0} + \frac{\ln(1/\varepsilon^2)}{\ln(1/\varepsilon)} = 2$ so $d = 2$

(by squeezing)

C)



K_∞ middle-thirds Cantor set.

$b_1 = 1/3$ $\leftarrow N(b_1) = 2$

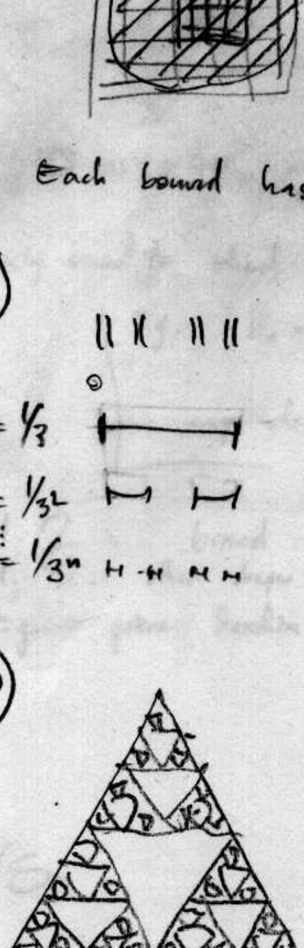
$b_2 = 1/3^2$ $\leftarrow N(b_2) = 2^2$

$b_n = 1/3^n$ $N(b_n) = 2^n$

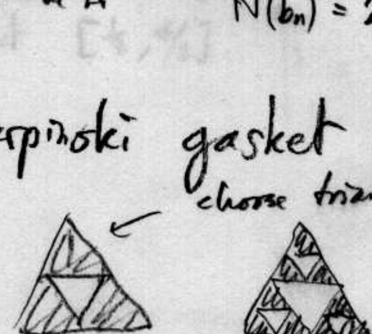
$$d = \lim_{n\to\infty}\frac{\ln N(b_n)}{\ln(1/b_n)} = \lim_{n\to\infty}\frac{n\ln 2}{n\ln 3} = \frac{\ln 2}{\ln 3} \approx 0.69\ldots$$

D)

Sierpinski gasket



choose triangular boxes



$b_1 = 1/2$ $\quad$ $b_2 = \frac{1}{2^2}$ ...

$N(b_1) = 3$ $\quad$ $N(b_2) = 3^2$

$$d = \lim_{n\to\infty}\frac{\ln(3^n)}{\ln(2^n)} = \frac{\ln 3}{\ln 2} \approx 1.58\ldots$$

E) $\frac{2\ln 2}{\ln 3}$