

SOLUTIONS

Math 53: Chaos! 2015: Midterm 1

2 hours, 50 points total, 4 questions, points somewhat  $\propto$  blank space. Good luck!

1. [11 points] Consider the map  $f(\mathbf{x}) = \begin{bmatrix} 1/2 & 0 \\ -1 & -1/2 \end{bmatrix} \mathbf{x}$ , where  $\mathbf{x} \in \mathbb{R}^2$ .
4. (a) Describe the object to which the unit disc maps under  $f$ . Include all relevant length(s) of the object. You do *not* have to give directions.

Ellipse with semiaxes given by  $\sqrt{\lambda(AA^T)}$



$$AA^T = \begin{bmatrix} 1/2 & 0 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/2 \\ -1/2 & 1/4 \end{bmatrix} \text{ so } \lambda^2 - (\frac{1}{4} + \frac{1}{4})\lambda + \frac{5}{16} - \frac{1}{4} = 0$$

$$\Rightarrow \lambda = \frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{16}} = \frac{3}{4} \pm \frac{1}{4} \quad \text{Semiaxes: } \sqrt{\frac{3}{4} + \frac{1}{12}}$$

$$\ell \sqrt{\frac{3}{4} - \frac{1}{12}}$$

1. (b) What is the area of the object you just described? [Reminder: unit disc has area  $\pi$ ].

$$\pi \cdot (\text{product of semiaxes}) \quad \text{or} \quad \pi \cdot \det(A) = \left| \begin{bmatrix} 1/2 & 0 \\ -1 & -1/2 \end{bmatrix} \right| = 1/4$$

$$\Rightarrow \pi/4$$

1. (c) Do any points from inside the unit disc get mapped outside the unit disc by  $f$ ?

Yes, since  $\sqrt{\lambda_1} = \sqrt{\frac{3}{4} + \frac{1}{12}} = \frac{1+\sqrt{2}}{2} > 1$  at any rate

An example one of you gave was  $f(0) = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$ , outside disc.

2. (d) What is the fate (asymptotic behavior) of orbits launched from all points  $\mathbf{x} \in \mathbb{R}^2$ , and why?

Since  $A$  triangular, can read eigenvalues  $\lambda(A)$  off diagonal:  
 they are  $\pm 1/2$ , so all  $|\lambda(A)| < 1$ ,  $\vec{0}$  is a sink,  
 and all orbits have  $\vec{0}$  as their limit. (Since linear map, basin is  $(\mathbb{R}^2)$ )

2. (e) Imagine the map is changed to  $f\left(\begin{array}{c} x \\ y \end{array}\right) = A\left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} x^2 + xy \\ y^3 \end{array}\right)$ . Classify the fixed point  $(0, 0)$  as a sink/source/saddle.

$$\text{Jacobi matrix } Df = A + \begin{bmatrix} \frac{\partial}{\partial x}(x^2 + xy) & \frac{\partial}{\partial y}(x^2 + xy) \\ \frac{\partial}{\partial x}y^3 & \frac{\partial}{\partial y}y^3 \end{bmatrix} \Bigg|_{\substack{x=0 \\ y=0}} = A + \begin{bmatrix} 2x+y & x \\ 0 & 3y^2 \end{bmatrix} \Bigg|_{\substack{x=0 \\ y=0}}$$

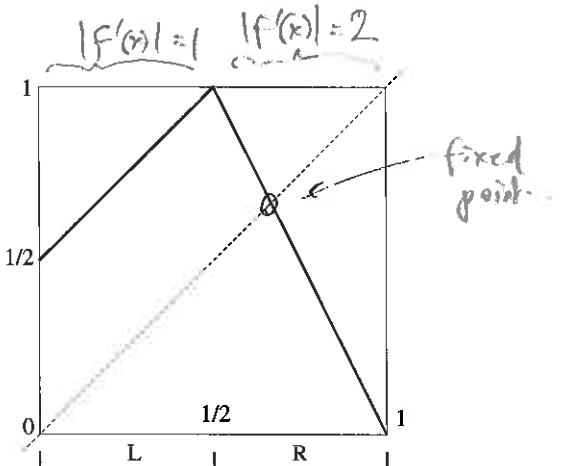
$\Rightarrow A$ , so eigenvalues same as before, so it's a sink.

2. [16 points]

Consider the function

$$f(x) = \begin{cases} x + 1/2, & x < 1/2 \\ 2(1-x), & x \geq 1/2 \end{cases}$$

whose graph is shown to the right, mapping  $[0, 1]$  to itself.



2. (a) Find and classify the fixed point(s).

See a fixed point where  $f(x) = x$  is where  $f(x) = 2(1-x)$   
 $\Rightarrow 2(1-x) = x$ ,  $x = \frac{2}{3}$ . Here  $|f'(\frac{2}{3})| > 1 \Rightarrow$  source.

3. (b) Find the Lyapunov exponent of the orbit starting at  $x = 1/3$ .

$$f\left(\frac{1}{3}\right) = \frac{5}{6}, \quad f\left(\frac{5}{6}\right) = 2\left(1 - \frac{5}{6}\right) = \frac{1}{3} \Rightarrow \text{period-2.}$$

$$\begin{aligned} \text{Lyapunov } h &= \frac{1}{2} [\ln |f'(p_1)| + \ln |f'(p_2)|] = \frac{1}{2} [\ln 1 + \ln 2] \\ &= \frac{1}{2} \ln 2 > 0 \end{aligned}$$

2. (c) Draw the transition graph for the intervals  $L$  and  $R$  shown.



since  $f(L) \supseteq R$   
 $f(R) \supseteq \{L \cup R\}$

- (d) Give a specific  $x$  which is eventually periodic but not periodic.

1. Find any pre-image of your known period-1 or period-2.

$$\text{Eg } x_0 = \frac{1}{6}, f(x_0) = \frac{2}{3}, f^k(x_0) = \frac{2}{3} \text{, } \forall k \geq 1$$

2. (e) Prove that there exists orbits which are not fixed, periodic, or eventually periodic.

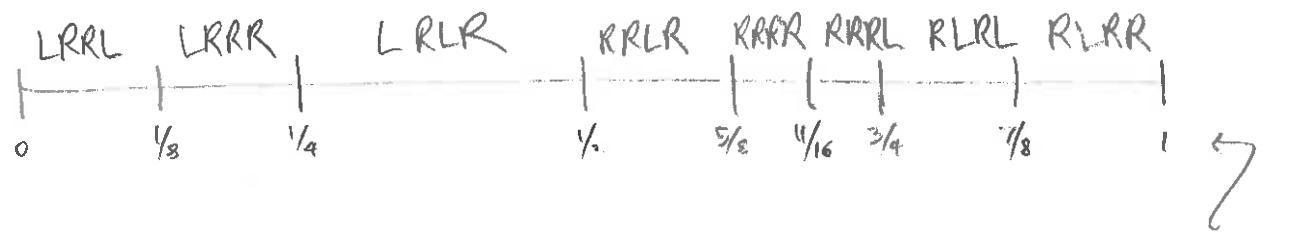
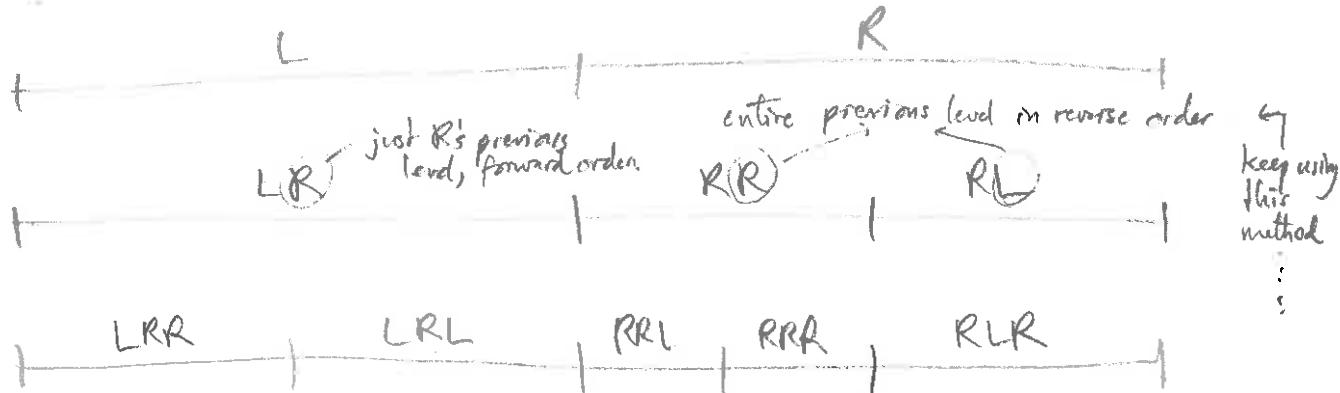
I merely need to construct a single example:

R, L, RRL, RRRL, RRRRL, ...

- (f) To what subinterval does the subinterval LRR map under one application of  $f$ ?

$$f(LRR) = RR, \text{ removes first letter}$$

5. (g) Show the subintervals down to level 4, with their correct ordering. Take plenty of horizontal space.



locations/sizes  
not graded,  
just for credit.

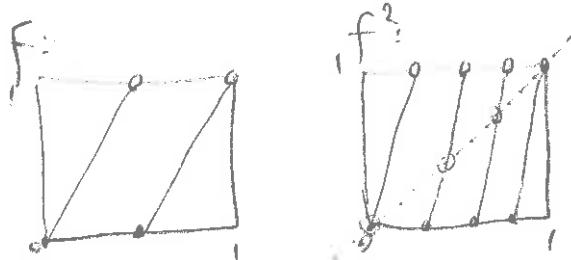
BONUS Prove a positive lower bound on the Lyapunov exponent of all orbits which never hit  $1/2$ .

- +1. Any orbit can only spend at most  $\frac{1}{2}$  the time in  $L$ , where  $|f'|=1$ ; the rest is in  $R$  where  $|f'|=2$

$$h = \lim_{n \rightarrow \infty} \sum_{k=1}^n \ln |f'(x_k)| \geq \frac{1}{2} \ln 2 \quad , \text{ same answer as (b).}$$

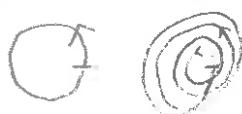
3. [15 points] Consider the map  $f(x) = 2x \pmod{1}$  mapping the (periodic) interval  $[0, 1)$  to itself.

- (a) Sketch a graph of  $f^2$ . How many fixed points of  $f^2$  are there in  $[0, 1)$ ?



3 fixed points since  
 $x=1$  is same as  $x=0$ .

- (b) Give the formula for the number of fixed points of  $f^k$ , for general  $k \geq 1$ .



'tortoise' goes round track once while 'hare' goes around  
 $y=x$   $y=2^k x$   $2^k$  times

$$\Rightarrow \# \text{ f.p.} \approx 2^k - 1$$

- (c) Work out the first 4 rows of the "periodic table" for the map, which computes how many periodic orbits there are of periods 1 through 4:

$k$	# f.p. of $f^k$	# accounted for by lower periods	# P.O.s period $k$
1	1	0	1
2	3	1	$1 = 2/2$
3	7	1	$2 = 6/3$
4	15	3	$3 = 12/4$

2. (d) Give the mathematical definition of what it means for a given point  $x_0$  to have *sensitive dependence*.

For any  $\varepsilon > 0$ , there is a point  $y_0$  such that  $|y_0 - x_0| < \varepsilon$  but for some  $k \geq 1$ ,  $|f^k(y_0) - f^k(x_0)| > d$ .

Here  $d$  is some fixed constant independent of  $\varepsilon$ .  
Usually  $O(1)$ .

3. (e) Prove that each point  $x_0 \in [0, 1]$  has sensitive dependence.

$$\text{Let } L = [0, \frac{1}{2}], R = [\frac{1}{2}, 1]$$

Subintervals are all size  $\frac{1}{2^k}$  at level  $k$ .

Transition graph is complete, so any itinerary allowed.

Let  $\varepsilon > 0$ , choose  $k > \log_2 \frac{1}{\varepsilon}$

Let  $J = S_1 \dots S_k$  be whatever  $x_0$ 's itinerary first  $k$  letters are. Pick  $y_0$  by reading off next 2 letters & adjusting as follows  
(four cases):

$$\begin{array}{ccc} J & \xrightarrow{\text{pick any } y_0 \text{ with}} & RR \\ LL & \xrightarrow{\quad} & \\ LR & \xrightarrow{\quad} & RR \\ RL & \xrightarrow{\quad} & LL \\ RR & \xrightarrow{\quad} & LL \end{array}$$

then after  $k$  iterations,  
 $|y_k - x_k| > \frac{1}{4}$  by  
the choice of 2-letters.

2. (f) Prove that each rational  $x_0$  is eventually periodic.

Let  $x_0 = p/q$ , then  $f(x_0) = x_1 = \frac{2p \bmod q}{q}$

$$x_2 = \frac{2^2 p \bmod q}{q} \quad \text{etc.}$$

There are only  $q$  places for  $x_k$  to go,  $\{0, \frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}\}$

so it must hit one it's already hit after  $\leq q$  iterations.

Then it must repeat.  $\Rightarrow EP$   
(since deterministic)

4. [8 points] Unrelated short questions. Please explain only briefly.
2. (a) What precisely are plotted on the horizontal and vertical axes of a bifurcation diagram?



horiz: some parameter in the map  
 vert: a large number of iterates  $x_n$ ,  
 usually excluding the first 100 or so  
 to allow settling into any attractor,  
 if there is one

2. (b) Give the mathematical definition of a fixed point  $p$  being sink.

There is an  $\varepsilon > 0$  such that for all  $x \in N_\varepsilon(p)$ ,

$$\lim_{k \rightarrow \infty} f^k(x) = p$$

2. (c) What set comprises the unstable manifold for the saddle at  $(0, 0)$  for the map  $f(x, y) = (0.9x, -1.1y)$  in  $\mathbb{R}^2$ ?

Diagonal matrix  $f[y] = A[y]$  for  $A = \begin{bmatrix} 0.9 & 0 \\ 0 & -1.1 \end{bmatrix}$   
 so  $x_k = (0.9)^k x_0$        $\cup$  manifold =  $\{(x, y) \in \mathbb{R}^2 : x = 0\}$ ,  
 $y_k = (-1.1)^k y_0$       the y-axis.

Since, unless  $x_0 = 0$ ,  $f^k$  will cause  $|x_k| \rightarrow \infty$  as  $k \rightarrow \infty$ .

2. (d) Estimate how many iterations of the map  $f(x) = 3x \pmod{1}$  it takes for a computer rounding error of around  $10^{-16}$  in the initial condition to cause an  $O(1)$  change in the iterate. (You can leave an expression if you like.)

Lyapunov  $h = \ln 3$ , or separation grows like  $3^k$

$$\text{So, separation } 1 \approx (10^{-16}) \cdot 3^k \quad \downarrow \ln$$

$$k = \frac{\ln 10^{16}}{\ln 3} \approx \frac{\ln 3^{32}}{\ln 3} \approx 32$$

(actually  $\approx 33.5$ )