Math 53 Fall 2013

Chaos!

First Midterm Exam

Tuesday, October 8, 5:00-7:00 PM

Your name (please print): ______________________________

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

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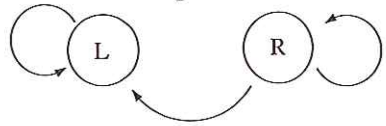
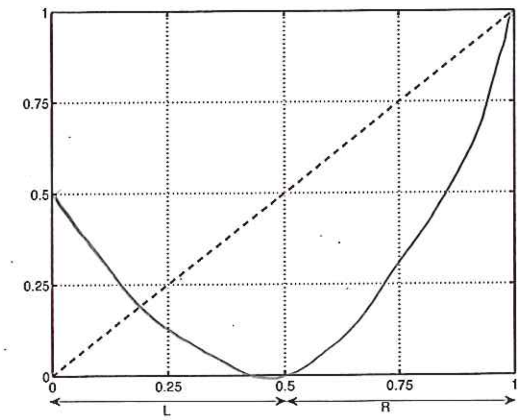
For grader use only:

Problem	Points	Score
1	15	
2	18	
3	15	
4	18	
5	16	
6	18	
Total	100	

1. (a) Using the axes below draw a possible smooth function $f(x)$ that has the following transition map.





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(b) What interval does the subinterval RL get mapped to?

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$f(RL) \in L$

(c) Give a list of *all* types of itineraries that *must* occur given the transition map.

$R^n\bar{L}$ $\quad n=0,1,2,\ldots$ 5

$\bar{L}$

2. Let $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5x \\ 2y - x^3 \end{bmatrix}$. The inverse of F is given by $F^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 0.5y + 4x^3 \end{bmatrix}$.
The origin is a saddle point of F.

(a) Show that the curve $y = \frac{8}{15}x^3$ is invariant under F.

We need to show $F\begin{pmatrix} x \\ 8/15x^3 \end{pmatrix}$ lies on the curve.

6

$$F\begin{pmatrix} x \\ 8/15x^3 \end{pmatrix} = \begin{pmatrix} 0.5x \\ \frac{16}{15}x^3 - x^3 \end{pmatrix} = \begin{pmatrix} 0.5x \\ \frac{1}{15}x^3 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} x/2 \\ \frac{8}{15}(x/2)^3 \end{pmatrix} \checkmark$$

(b) Show that the stable manifold is given by $y = \frac{8}{15}x^3$.

$$F^n(\bar{x}) = \begin{pmatrix} x/2^n \\ \frac{8}{15}x^3 2^{-3n} \end{pmatrix}$$

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$$\lim_{n\to\infty} F^n(\bar{x}) = \lim_{n\to\infty} \begin{pmatrix} x/2^n \\ \frac{8}{15}x^3 2^{-3n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \checkmark$$

$\therefore$ this is the stable manifold.

(c) Show that the unstable manifold is given by $x = 0$.

By defn., We need to show $\lim_{n\to\infty} F^{-n}\begin{pmatrix} 0 \\ y \end{pmatrix} = \bar{0}$

6

$$F^{-n}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y/2^n \end{pmatrix}$$

$$\therefore \lim_{n\to\infty} F^{-n}\begin{pmatrix} 0 \\ y \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} 0 \\ y/2^n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

6. Consider the map $f(x) = 2x(\text{mod } 1)$ on the interval $[0, 1)$.

(a) Draw $f(x)$ and $f^2(x)$.

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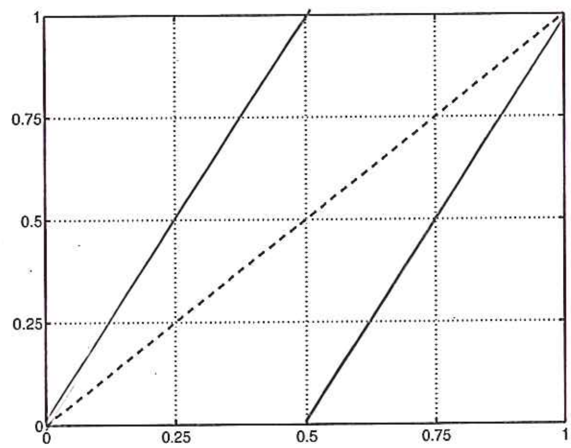
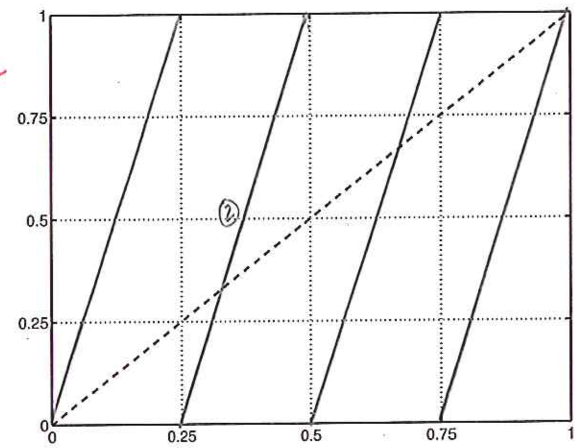


$f(x)$

2



$f^2(x)$

(b) Find the fixed points of $f(x)$.

$x = 0.$

2

(c) How many fixed points does $f^2(x)$ have?

3.

2

(d) How many 2-periodic orbits are there? Find the points in the orbit(s).

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1. 2-periodic orbit.

We need to find 1 of the pts. we know line ②.

has slope 4 & goes through pt $(0.25, 0)$

line is $y = 4 \cdot (x - 0.25) = 4x - 1$.

find where it intersects w/ $y = x$. $\Rightarrow x = 4x - 1$

$\rightarrow x = 1/3$.

$\Rightarrow P_1 = 1/3$

$P_2 = f(1/3) = 2/3$

$\{1/3, 2/3\}$ is the orbit.

(e) Prove that if the orbit $\{x_0, x_1, \ldots\}$ is eventually periodic then x_0 is a rational number.

Assume $\{x_0, x_1, \ldots\}$ is eventually periodic $\Rightarrow$

$\Rightarrow$ for some positive N $f^{n+p}(x_0) = f^n(x_0)$ $\forall n \geq N$

$\Rightarrow$ $2^{n+p}x_0 = 2^n x_0 + m \pmod 1$ m an integer.

$\Rightarrow 2^n x_0 (2^p - 1) = m$

$\Rightarrow$ $x_0 = \dfrac{m}{2^n(2^p-1)}$ $\Rightarrow$ x_0 is a rational #.

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BONUS: How many 3-periodic orbits are there? Find the points in the orbit(s).

There are 7 fixed pts 8 of $f^3(x)$

1 is the fixed pt of $f(x)$.

$\Rightarrow$ 2 3-periodic orbits.

+3.

The slope of all lines is 8.

The 2nd line goes through $(1/8, 0)$

$\Rightarrow$ $y = 8(x - 1/8)$ is the equation of the line

$\Rightarrow$ 1st fixed pt is at $8x - 1 = x$ $\Rightarrow x = 1/7$

$P_1 = 1/7$ $P_2 = 2/7$ $P_3 = 4/7$

1st orbit is $\{1/7, 2/7, 4/7\}$

Next line goes through $(7/8, 0)$

$y = 8(x - 3/8)$ is the next line.

next fixed pt is when $x = 8x - 3$

$\Rightarrow x = 3/7$

$P_2 = f(3/7) = 6/7$ $P_3 = f(6/7) = 12/7 = 5/7$

$\{3/7, 6/7, 5/7\}$

3. Consider the Hénon map with $a = 0.16$ and $b = 0.4$,

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.16 - x^2 + 0.4y \\ x \end{bmatrix}.$$

Find and classify the fixed points of the map.

Fixed pts satisfy $\begin{bmatrix} .16 - x^2 + 0.4y \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow y = x \qquad 0.16 - x^2 + 0.4x = x$

$\Rightarrow x^2 + 0.6x - 0.16 = 0.$

$\Rightarrow x = \frac{-0.6 \pm \sqrt{.36 + 0.64}}{2} = \frac{-0.6 \pm 1}{2}$

$= \frac{-1.6}{2}, \frac{+0.4}{2} = -0.8, +0.2$

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to Classify look at eigenvalues of the Jacobian.

1.6×1.6: 96, 160, 2.56

$Df(x) = \begin{bmatrix} -2x & 0.4 \\ 1 & 0 \end{bmatrix}$

$Df(0.8) = \begin{bmatrix} 1.6 & 0.4 \\ 1 & 0 \end{bmatrix}$, eigenvalues $\begin{vmatrix} 1.6-\lambda & 0.4 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1.6-\lambda) - 0.4 = 0$

$\Rightarrow \lambda^2 - 1.6\lambda - 0.4 = 0$

$\lambda = \frac{-1.6 \pm \sqrt{1.6^2 - 4(0.4)}}{2}$

$= \frac{-1.6 \pm \sqrt{2.56 - 1.6}}{2}$

$= \frac{-1.6 \pm \sqrt{4.16}}{2}$

$1.6 + \sqrt{4.16} > 2 \Rightarrow \left|\frac{-1.6 + \sqrt{4.16}}{2}\right| > 1$

$\left|\frac{1.6 - \sqrt{4.16}}{2}\right| < 1.$

$(0.8, 0.8)$ is a saddle.

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$$Df(-0.2) = \begin{bmatrix} -0.4 & 0.4 \\ 1 & 0 \end{bmatrix}$$

eigenvalues $\begin{vmatrix} -0.4-\lambda & 0.4 \\ 1 & -\lambda \end{vmatrix} = -\lambda(-0.4-\lambda) - 0.4 = \lambda^2 + 0.4\lambda - 0.4 = 0$

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$$\lambda = \frac{-0.4 \pm \sqrt{0.16 - 4(-0.4)}}{2} = -0.2 \pm \frac{1}{2}\underbrace{\sqrt{1.76}}_{< 0.8}$$

$|\lambda| < 1 \Rightarrow (-0.2, 0.2)$ is a sink.

4. Consider $\mathsf{A} = \begin{bmatrix} \frac{1}{2} & 0 \\ 2 & \frac{1}{2} \end{bmatrix}$.

(a) Find the length of semi-axes of the ellipse that the unit ball $\{x \in \mathbb{R}^2 : \|x\| = 1$ gets mapped to under A. [Hint: it may be helpful to factor out $\frac{1}{4}$.] What is the closest distance to the origin? What is the furthest distance from the origin?

$$AA^T = \begin{bmatrix} 1/2 & 0 \\ 2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 1 \\ 1 & 17/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 4 \\ 4 & 17 \end{bmatrix}.$$

now find eigenvalues.

$$\begin{vmatrix} 1-\lambda & 4 \\ 4 & 17-\lambda \end{vmatrix} = (1-\lambda)(17-\lambda) - 16$$

$$= 17 - 18\lambda + \lambda^2 - 16$$

$$= \lambda^2 - 18\lambda + 1 = 0$$

$$\lambda = \frac{18 \pm \sqrt{18^2 - 4 \cdot 1}}{2} = \frac{18 \pm \sqrt{320}}{2}$$

eigenvalues are

$$\lambda_1 = \frac{18 + \sqrt{320}}{8} \quad \lambda_2 = \frac{18 - \sqrt{320}}{8}.$$

$\sqrt{\lambda_2}$ is closest to origin

$\sqrt{\lambda_1}$ is furthest from origin.

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(b) What are all the possible fates of orbits starting within or on the unit circle? (i.e., where do the orbits tend after repeated iterations?)

note $\sqrt{\lambda_2} > 1 \rightarrow$ some points will be mapped outside of unit circle. on lower iterations but since the eigenvalues of A have magnitude <1. all orbits will eventually converge to origin.

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(c) What is the area enclosed by the image of the unit circle under one application of A?

3

Area $= |\det A| \pi = \frac{1}{4}\pi$

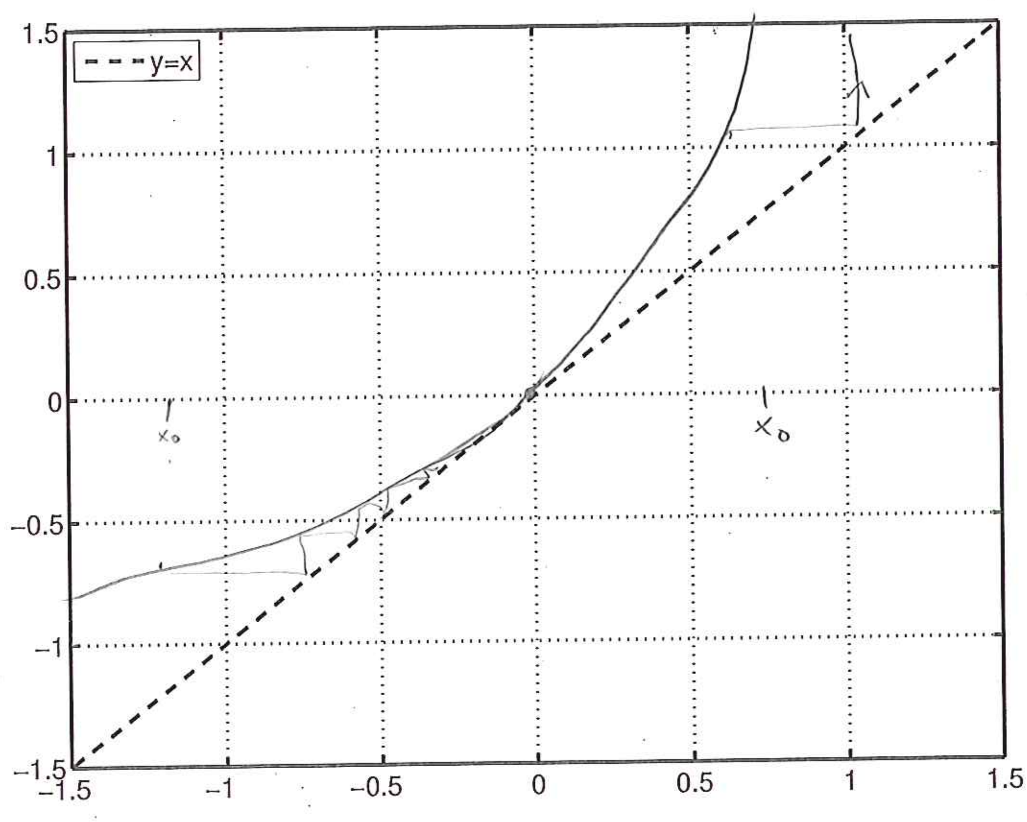
5. Consider the map $f(x) = e^x - 1$.

(a) Find the fixed points and classify them as attracting, repelling or neither. If there is a sink, find the basin of attraction.

fixed pt is $x = 0$.

$x=0$. is a sink with a basin of attraction of $\{x : x \leq 0\}$.

6



(b) Write down the precise definition of a one-dimensional source.

a fixed pt p of $f(x)$ is a source.
iff there is an epsilon neighborhood of p $N_\varepsilon(p)$
st each $x \in N_\varepsilon(p)$ (except p itself) eventually
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maps outside of $N_\varepsilon(p)$.

(c) Write down the precise definition of an eventually periodic orbit.

x_0 is eventually periodic with period
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p for f if for some positive N.
$f^{n+p}(x_0) = f^n(x_0)$ $\forall n \geq N$. & if p is the
smallest such positive integer.