# Math 53 Fall 2013 

## Chaos!

# First Midterm Exam 

Tuesday, October 8, 5:00-7:00 PM

Your name (please print): $\qquad$

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

For grader use only:

| Problem | Points | Score |
| :---: | ---: | :--- |
| 1 | 15 |  |
| 2 | 18 |  |
| 3 | 15 |  |
| 4 | 18 |  |
| 5 | 16 |  |
| 6 | 18 |  |
| Total | 100 |  |

1. (a) Using the axes below draw a possible smooth function $f(x)$ that has the following transition map.


(b) What interval does the subinterval $R L$ get mapped to?
(c) Give a list of all types of itineraries that must occur given the transition map.
2. Let $\boldsymbol{F}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{c}0.5 x \\ 2 y-x^{3}\end{array}\right]$. The inverse of $\boldsymbol{F}$ is given by $\boldsymbol{F}^{-1}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{c}2 x \\ 0.5 y+x^{3}\end{array}\right]$. The origin is a saddle point of $\boldsymbol{F}$.
(a) Show that the curve $y=\frac{8}{15} x^{3}$ is invariant under $\boldsymbol{F}$.
(b) Show that the stable manifold is given by $y=\frac{8}{15} x^{3}$.
(c) Show that the unstable manifold is given by $x=0$.
3. Consider the Hénon map with $a=0.16$ and $b=0.4$,

$$
\boldsymbol{F}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{c}
0.16-x^{2}+0.4 y \\
x
\end{array}\right]
$$

Find and classify the fixed points of the map.
4. Consider $A=\left[\begin{array}{cc}\frac{1}{2} & 0 \\ 2 & \frac{1}{2}\end{array}\right]$.
(a) Find the length of semi-axes of the ellipse that the unit ball $\left\{\boldsymbol{x} \in \mathbb{R}^{2}:\|\boldsymbol{x}\|=1\right.$ gets mapped to under A. [Hint: it may be helpful to factor out $\frac{1}{4}$.] What is the closest distance to the origin? What is the furthest distance from the origin?
(b) What are all the possible fates of orbits starting within or on the unit circle? (i.e., where do the orbits tend after repeated iterations?)
(c) What is the area enclosed by the image of the unit circle under one application of $A$ ?
5. Consider the map $f(x)=e^{x}-1$.
(a) Find the fixed points and classify them as attracting, repelling or neither. If there is a sink, find the basin of attraction.

(b) Write down the precise definition of a one-dimensional source.
(c) Write down the precise definition of an eventually periodic orbit.
6. Consider the map $f(x)=2 x(\bmod 1)$ on the interval $[0,1)$.
(a) Draw $f(x)$ and $f^{2}(x)$.


(b) Find the fixed points of $f(x)$.
(c) How many fixed points does $f^{2}(x)$ have?
(d) How many 2-periodic orbits are there? Find the points in the orbit(s).
(e) Prove that if the orbit $\left\{x_{0}, x_{1}, \ldots\right\}$ is eventually periodic then $x_{0}$ is a rational number.

BONUS: How many 3-periodic orbits are there? Find the points in the orbit(s).

