Worksheet #7: Linear torus map

Consider the map $f(\boldsymbol{x}) = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \pmod 1$, where a, b, c, and d are integers.

(1) Assume A has no eigenvalue equal to 1. Write down a condition on a, b, c, and d such that this is true.

need $\begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix} \neq 0 \Rightarrow (a-1)(d-1) - cd \neq 0.$

This is needed for a fixed pt to exist

(2) Show that $f(\boldsymbol{p}) = \boldsymbol{p}$ implies $\boldsymbol{p}$ has rational components $\begin{bmatrix} x \\ y \end{bmatrix}$.

$f(p)=p \rightarrow \begin{aligned} ax+by &= x+n \\ cx+dy &= y+m \end{aligned}$ $m, n \in \mathbb{Z}$ needed for mod 1 condition

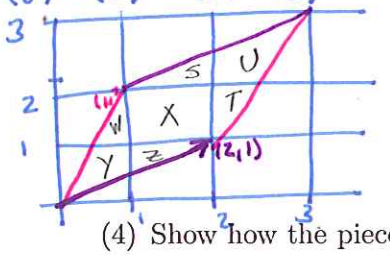
$\rightarrow \begin{aligned} (a-1)x+by &= n \\ cx+(d-1)y &= m \end{aligned} \Big\} \rightarrow \begin{aligned} c(a-1)x+cby &= cn \\ \ominus\ (a-1)cx+(a-1)(d-1)y &= (a-1)m \end{aligned}$

$(cb-(d-1)(a-1))y = cn-(a-1)m$

$\neq 0$. by (1) $\Rightarrow y = \frac{\text{integer}}{\text{integer}} = \text{rational}$

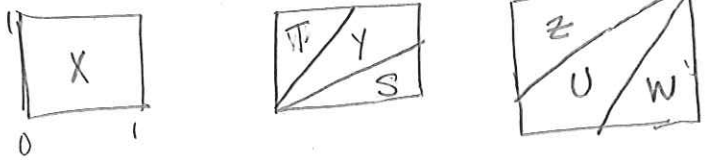
(3) Draw the action of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ on the unit square.

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$ $A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$



(4) Show how the pieces rearrange to fill some squares.

Area(Y) = Area(U)
Area(T) = Area(W)
Area(S) = Area(Z)



3 Unit squares!

(5) How many squares are filled for a general A?

$|\det A|$ ie $\det A$ gives area expansion factor.

(6) How many solutions are there to $f(\boldsymbol{x}) = \boldsymbol{x_0}$ for a given $\boldsymbol{x_0} \in \Pi^2$.

Since $|\det A|$ = # of filled squares = 3.
A distinct solution exist for each sq $\Rightarrow$ 3 solutions.

(7) BONUS: How many solutions to $f(\boldsymbol{x}) = \boldsymbol{x}$ are there? [Hint: use matrix A − I from above.

$f(x) - x = 0$ ie $(A-I)\bar{x} = 0$. $\bar{x}$ x_0. So there are

$|\det(A-I)|$ fixed pts.