

Worksheet #7: Linear torus map

Consider the map $\mathbf{f}(\mathbf{x}) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \pmod{1} = \mathbf{A}\mathbf{x} \pmod{1}$, where $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and a, b, c , and d are integers.

(1) Assume \mathbf{A} has no eigenvalue equal to 1. Write down a condition on a, b, c , and d such that this is true.

(2) Show that $\mathbf{f}(\mathbf{p}) = \mathbf{p}$ implies \mathbf{p} has rational components $\begin{bmatrix} x \\ y \end{bmatrix}$.

(3) Draw the action of $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ on the unit square.

(4) Show how the pieces rearrange to fill some squares.

(5) How many squares are filled for a general \mathbf{A} ?

(6) How many solutions are there to $\mathbf{f}(\mathbf{x}) = \mathbf{x}_0$ for a given $\mathbf{x}_0 \in \Pi^2$.

(7) BONUS: How many solutions to $\mathbf{f}(\mathbf{x}) = \mathbf{x}$ are there? [Hint: use matrix $\mathbf{A} - \mathbf{I}$ from above.]