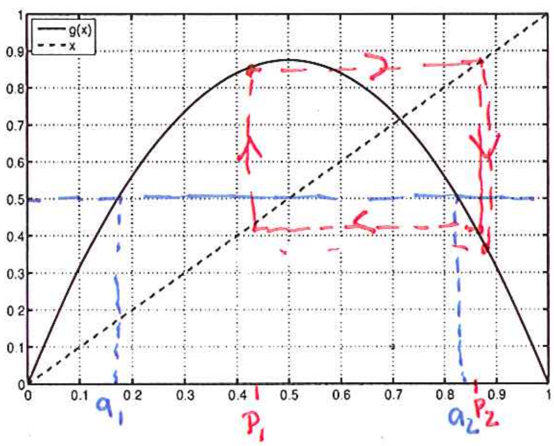
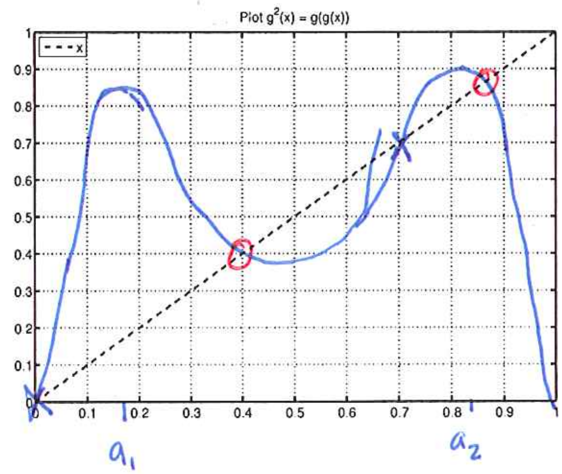
Worksheet #2: Periodic sinks and sources

Consider the function $g(x) = \frac{7}{2}x(1-x)$ ie. a logistic function with $a = \frac{7}{2}$. $x = \frac{5}{7}$ is a fixed point of $g^2(x)$.





(1) Is there a 2-periodic orbit of g? If so, what is the orbit?

need to find fixed pts of $g^2(x) = g(g(x)) = g\left(\frac{7}{2}x(1-x)\right) = \left(\frac{7}{2}\right)^2 x(1-x)\left(1 - \frac{7}{2}x(1-x)\right)$

$= \frac{-343x^4}{8} + \frac{343x^3}{4} - \frac{441}{8}x^2 + \frac{49x}{4}$

fixed pts are $x = 0$, $x = 5/7$, $x = 3/7$, $x = 6/7$ 2-period orbit is $\left\{\frac{3}{7}, \frac{6}{7}\right\}$

(2) How many fixed points does g^2 have, at least?

At least 4 fixed pts.

(3) Is $p_1 = \frac{3}{7}$ a periodic sink, source or can you not tell? $g'(x) = \frac{7}{2}(1-2x)$

$|g^{(2)\prime}(p_1)| = |g'(p_1)\, g'(p_2)| = \left(\frac{7}{2}\right)^2 \left|(1 - 6/7)(1 - 12/7)\right| = \left(\frac{7}{2}\right)^2 \left|\frac{1}{7}(-5/7)\right| = \frac{5}{4} > 1$

$\Rightarrow$ periodic source.

(4) Is p_2 also a period-2 sink, souce or can you not tell? Does this answer agree with p_1? Explain.

Yes, since $|g^{(2)\prime}(p_1)| = |g^{(2)\prime}(p_2)|$.
Since they are periodic, they must have same behavior.

(5) Generalize the derivative test: If $\{p_1, \ldots, p_k\}$ is a periodic-k orbit of f, what is $(f^k)'$ at $x = p_1$ in terms of f'? [Hint: Use induction.]

We know $|f^{(2)\prime}(p_1)| = |f'(p_1)\, f'(p_2)|$

etc. $|(f^{(k)})'(p_j)| = \left|\prod_{i=1}^{k} f'(p_i)\right|$

(6) Does the test care which p_i you evaluate $(f^k)'$ at?

No. since derivative is evaluated at all pts in the orbit.