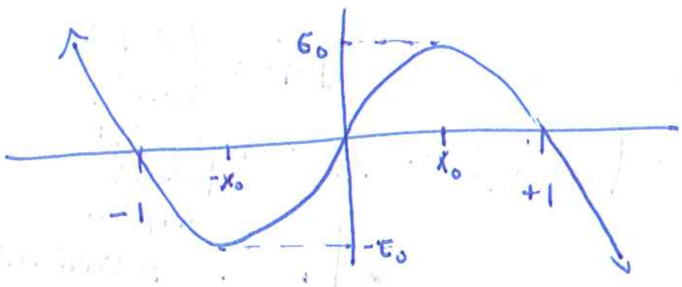
Worksheet #13: Motion in a potential field

Consider the differential equation $x'' + 1 - 3x^2 = 0$.

(1) What is $\frac{dP}{dx}$? What is $P(x)$?

$$\frac{dP}{dx} = 1 - 3x^2 \qquad P(x) = x - x^3$$

(2) Sketch $P(x)$.



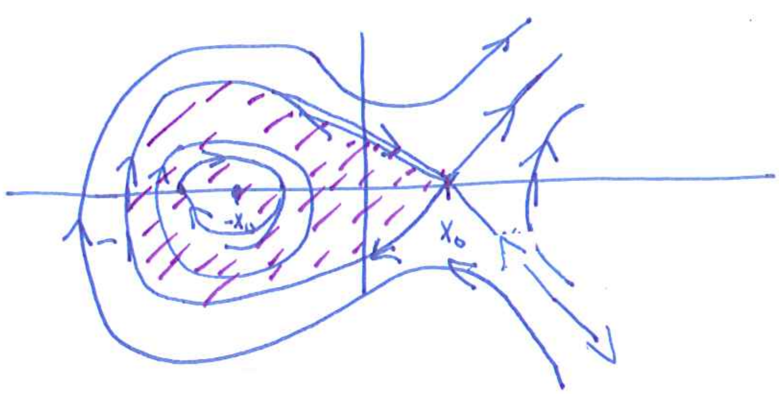
(3) Write the differential equation as a first order system.

$$x' = y$$
$$y' = -\frac{dP}{dx} = -1 + 3x^2 \qquad \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{pmatrix} y \\ -1 + 3x^2 \end{pmatrix}$$

(4) Find the equilibria.

$$y = 0$$
$$-1 + 3x^2 = 0 \rightarrow x = \pm\sqrt{1/3} = \pm\sqrt{3}/3$$
$$(x_0, y_0) = (\pm\sqrt{3}/3, 0)$$

(5) Sketch the level curves of $E(x, x') = 1/2(x')^2 + P(x)$ in the plane:



(6) What kinds of periodic orbits can happen? What range of energies E may they have?

oscillating periodic orbits around equilibrium $x_0 = -1/\sqrt{3}$ $E < E_0 = P(1/\sqrt{3}) = \frac{2}{3\sqrt{3}}$

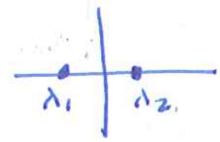
(7) When is the motion unbounded?

It is unbounded for $E > E_0$.

(8) Deduce the stability using the Jacobian Df at the equilibria. $Df = \begin{pmatrix} 0 & 1 \\ 6x & 0 \end{pmatrix}$

$x_0 = \frac{\sqrt{3}}{3}$ $Df(x_0) = \begin{pmatrix} 0 & 1 \\ 2\sqrt{3} & 0 \end{pmatrix} \Rightarrow \lambda = \pm\sqrt{2\sqrt{3}}$

λs pure



$\Rightarrow$ unstable.

$x_0 = -\frac{\sqrt{3}}{3}$ $Df(x_0) = \begin{pmatrix} 0 & 1 \\ -2\sqrt{3} & 0 \end{pmatrix} \Rightarrow \lambda = \pm i\sqrt{2\sqrt{3}}$

$Re(\lambda) = 0$ but nonlinear thm doesnot give stability

fortunately definition of stability Does. (ie all orbits near by stay in the stable region. (Shaded purple)