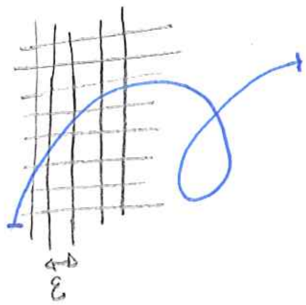
Worksheet #12: Box-counting dimension

Definition: $\text{boxdim(S)} = \lim_{\epsilon \to 0} \dfrac{\ln \text{N}(\epsilon)}{\ln(1/\epsilon)}$

Using the 3 simplications from class, find (and prove if you can) the box dimension for the following sets:

(1) Curve of length L. [Hint: Is there a rigous upper bound on the number of boxes the curve can touch? Consider breaking the curve into pieces of length ϵ.]



at most 4 boxes touch each ε length of curve

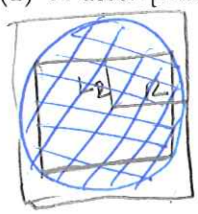
$\Rightarrow N(\varepsilon) \le 4\,L/\varepsilon$

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \le \lim_{\varepsilon \to 0} \frac{\ln\left(\frac{4L}{\varepsilon}\right)}{\ln(1/\varepsilon)}$$

$$= \lim_{\varepsilon \to 0} \underbrace{\frac{\ln 4L}{\ln(1/\varepsilon)}}_{\to 0} + 1 = 1$$

$\Rightarrow d \le 1.$

(2) A disc. [Hint: Is there a shape with which all boxes must lie?]

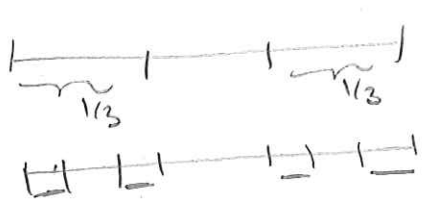


There are 2 squares with side length $L_1 = 2R$ & $L_2 < 2R$. st

$$\frac{L_2^2}{\varepsilon^2} \le N(\varepsilon) \le \frac{L_1^2}{\varepsilon^2}$$

$$d = \lim_{\varepsilon \to 0} \frac{\ln(L^2/\varepsilon^2)}{\ln(1/\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\ln(L^2)}{\ln(1/\varepsilon)} + \frac{\ln(1/\varepsilon^2)}{\ln(1/\varepsilon)} = 2$$

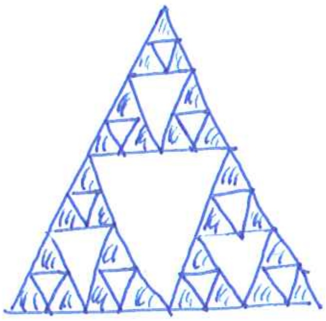
(3) K_∞- the middle third Cantor set.

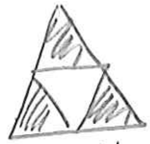


	$N(b)$	b
K_1	2	1/3
K_2	4	1/9
K_n	2^n	$(1/3)^n$

$$d = \lim_{n \to \infty} \frac{\ln N(b_n)}{\ln(1/b_n)} = \lim_{n \to \infty} \frac{n \ln 2}{n \ln 3} = \frac{\ln 2}{\ln 3} = 0.69$$

(4) Sierpinski Gasket





$b_1 = 1/2$

$N(b_1) = 3$

$b_2 = 1/4$

$N(b_2) = 9$

$b_n = (1/2)^n$

$N(b_n) = 3^n$

$$d = \lim_{n \to \infty} \frac{\ln(3^n)}{\ln(2^n)} = \frac{\ln 3}{\ln 2} = 1.58\ldots$$

(5) $K_\infty \times K_\infty \subset [0,1]^2$

$$\frac{2 \ln 2}{\ln 3}$$