Worksheet #11: Points in the Mandelbrot set

Definition: $M = \{c \in \mathbb{C} : 0 \text{ is not in basin of } \infty \text{ for the map } P_c(z) = z^2 + c\}$

(1) Is $c = -1$ in M?

lets iterate starting at 0. $P_c(0) = -1 \to (-1)^2 - 1 = 0 \to -1$ etc.

$\Rightarrow$ 2period orbit.

So it does not go to ∞. $\Rightarrow$ yes in M.

(2) Is $c = 1$ in M? How many iterations did you need before you believed this?

iterate $0 \to 1 \to 1^2 + 1 = 2 \to 5 \to 26 \to \infty$.

(5) exceeds 2 so its fate is sealed

Not in M.

(3) Is $c = i$ in M?

$0 \to i \to i^2 + i = -1 + i \to (-1+i)^2 + i = 1 - 2i - 1 + i = -i \to -1 + i$

$\Rightarrow$ 2period orbit.

(4) Check the *stability* of the orbit you just formed. (i.e., Is it a sink, source or saddle?)

[Hint: either use $\begin{bmatrix} x \\ y \end{bmatrix}$ map in $\mathbb{R}^2$ or cheat and use the 1D formula.]

$z = x + iy$

$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x^2 - y^2 + a \\ 2xy + b \end{bmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy + 1 \end{pmatrix}$ where $c = a + ib$. $c = i \Rightarrow a = 0 \ b = 1$

$Df(x) = \begin{bmatrix} -2x & -2y \\ 2y & 2x \end{bmatrix}$

$Df^2\begin{pmatrix} 0 \\ -1 \end{pmatrix} = Df\begin{pmatrix} -1 \\ 1 \end{pmatrix} Df\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix}$

eigenvalues $\lambda = -4 \pm 4i$

$|\lambda| = 4\sqrt{2} > 1. \Rightarrow$ unstable source.

(5) What does Fatou's theorem tell you about if another sink could exist?

Sinks must have zero in the basin so no sink exist.

(6) So what shape/size is $J(c)$ for $c = i$?

it must have zero measure but be connected since $i \in M$.

(7) Do you expect $c = i$ to be in the interior, boundary, or exterior of M? [Hint: perturb c]

It is on the boundary since a slight perturbation leads to falling off the unstable period-2 orbit.

(8) Find a simple linear conjugacy between $P_c(z) = z^2 + c$, where c, and z are real and the logistic function $g_a(x) = ax(1 - x)$, for some a related to c

Goal: find $h(x) = \alpha x + \beta$ st $g_a(h(x)) = h(P_c(x))$

$a(\alpha x + \beta)(1 - \alpha x - \beta) = \alpha x^2 + c + \beta$

$\Rightarrow a[\alpha x - \alpha^2 x^2 - \alpha\beta x + \beta - \alpha\beta x - \beta^2] = \alpha x^2 + c + \beta$

Collect like terms:

$1: \ a\beta - a\beta^2 = c + \beta \Rightarrow c = -\frac{\beta}{4}(2 - \beta)$

$x: \ a\alpha - 2a\beta\alpha = 0 \Rightarrow \beta = \frac{1}{2}$

$x^2: \ -a\alpha^2 = \alpha \Rightarrow \alpha = \frac{-1}{a}$