# Math 53 Fall 2013

# Chaos!

# Second Midterm Exam

Thursday, November 7, 5:00-7:00 PM

Your name (please print): ______________________________

**Instructions:** This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

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For grader use only:

| Problem | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 20 | |
| 3 | 25 | |
| 4 | 15 | |
| 5 | 15 | |
| Total | 100 | |

1. Consider the second order nonlinear differential equation $x'' + x^2 - 4 = 0$. In this question be careful about *signs*.

$x'' = -\frac{dP}{dx}$

(a) Write the equation as a first order system.

$y = x'$
$y' = -x^2 + 4$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} y \\ -x^2+4 \end{bmatrix}$$

(b) Find all equilibrium points and use linearization to learn some information about their stability.

$y = 0 \quad x = \pm 2$

$$Df = \begin{bmatrix} 0 & 1 \\ -2x & 0 \end{bmatrix}$$

$$Df(2,0) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$$

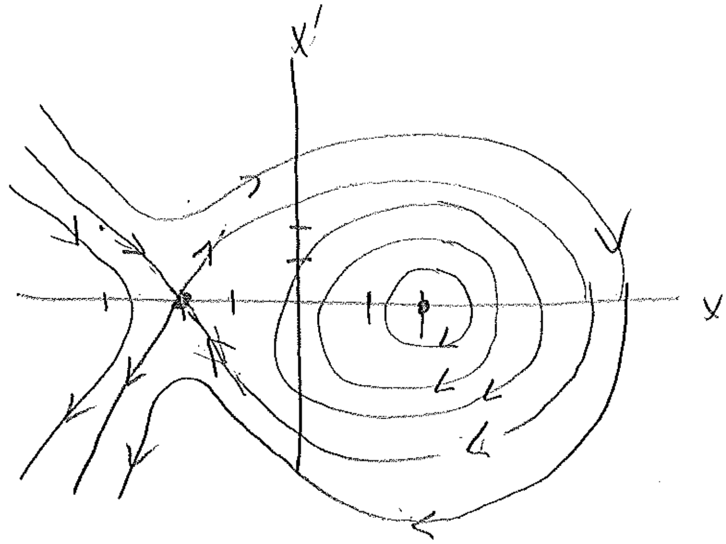
eigenvalues
$\lambda^2 + 4 = 0$
$\rightarrow \lambda = \pm 2i$
$\rightarrow$ Stable periodic equilibrium

$$Df(-2,0) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

eigenvalues
$\lambda^2 - 4 = 0$
$\lambda = \pm 2$

$\rightarrow$ Saddle pt.

(c) Sketch the phase plane $(x, x')$ showing several orbits including all types of motion that can occur.




eigenvectors for Saddle pt.

$$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$x_2 = 2x_1$

$\lambda_1 = 2 \quad \vec{V}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix}$$

$x_2 = -2x_1$

$\lambda_2 = -2 \quad \vec{V}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(d) What is the potential function $P(x)$ for the differential equation?

$$P(x) = \int x^2 - 4\,dx = \frac{x^3}{3} - 4x$$

roots $x=0,\ x = \pm \frac{2}{\sqrt{3}}$

(e) In what set of energies do periodic orbits lie? (Watch your endpoints.)

Periodic orbits occur when E is bounded between P(-2) & P(2).

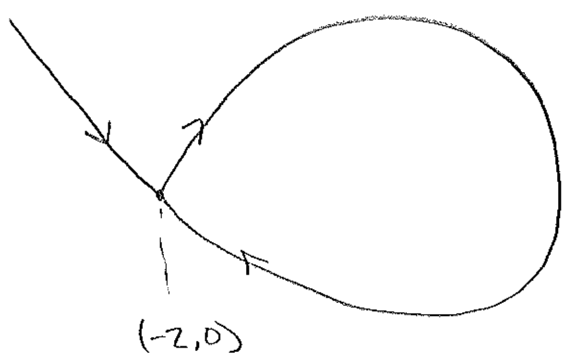
$8 - 2\frac{2}{3}$

$P(-2) = \frac{-8}{3} + 8 = 5\frac{1}{3} \quad P(2) = \frac{8}{3} - 8$

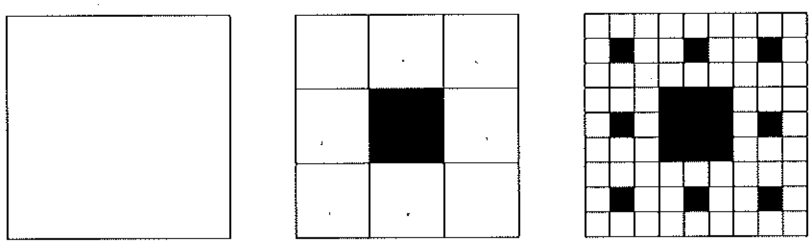
$-5\frac{1}{3} < E < 5\frac{1}{3}$

(f) Sketch the phase plane of all points that have an unstable equilibrium as their limit as $t \to \infty$.

This is the unstable manifold!



2. (a) Find the box-counting dimension of the subset of $\mathbb{R}^2$ formed by the limiting process sketched below: Taking a square partitioning it into 9 equi-sized squares and removing the middle square. In the figure, black boxes are not in the set while white boxes are in the set. [Hint: describe your boxes.]



$K_0$ $\qquad$ $K_1$ $\qquad$ $K_2$

$b_1 = 1/3 \quad N(b_1) = 8$

$b_2 = 1/9 \quad N(b_2) = 8 \cdot 8$

$\vdots$

$b_n = (1/3)^n \quad N(b_n) = 8^n$

$$\text{boxdim}(S) = \lim_{n\to\infty} \frac{\ln(N(b_n))}{\ln(1/b_n)} = \lim_{n\to\infty} \frac{n \ln 8}{n \ln 3} = \frac{\ln 8}{\ln 3}$$

(b) Show that the limiting set has 0 area.

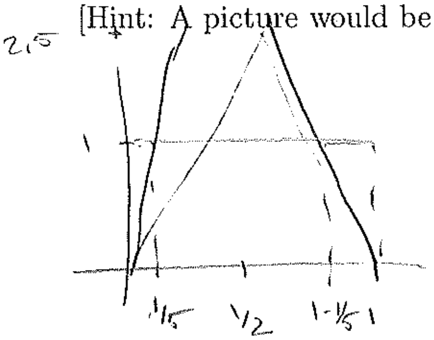
Area of each box $= ((1/3)^n)^2$ there are $8^n$ boxes.

$$\Rightarrow \text{Total area} = \lim_{n\to\infty} \frac{8^n}{9^n} = \lim_{n\to\infty} \left(\frac{8}{9}\right)^n = 0.$$

(c) Find the box counting dimension of the set of initial values whose orbits remain bounded for all iterations, under the map

$$T(x) = \begin{cases} 5x & x \le 1/2 \\ 5(1-x) & x \ge 1/2 \end{cases}$$

[Hint: A picture would be useful. Bonus: what type of set is the attractor?]




$b_1 = 1/5 \quad N(b_1) = 2.$

$b_2 = (1/5)^2 \quad N(b_2) = 4$

$\vdots$

$b_n = (1/5)^n \quad N(b_n) = 2^n.$

$$\text{boxdim}(S) = \lim_{n\to\infty} \frac{\ln(N(b_n))}{5\ln(1/b_n)} = \lim_{n\to\infty} \frac{n \ln 2}{n \ln 5} = \frac{\ln 2}{\ln 5}$$

Limiting set is a cantor set.

(d) What is the measure of the attracting set in (c)? [Bonus: prove it.]

Set of measure 0.

(e) Does the set in (c) contain a countable or uncountable number of points? Finite or infinite?

The set is uncountable & infinite.

3. Consider the following map acting on points $(x, y)$ in the unit square $[0,1]^2$:

$$B(x,y) = \begin{cases} \left(\frac{x}{9}, 2y(\text{mod } 1)\right) & \text{for } y > 1/2 \\ \left(\frac{x+8}{9}, 2y(\text{mod } 1)\right) & \text{for } y \leq 1/2 \end{cases}$$

(a) Compute the complete set of Lyapunov exponents (for almost all initial conditions) of this map? Explain reasoning.

$$Df(x) = \begin{bmatrix} 1/9 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow J_n = J^n = \begin{bmatrix} (1/9)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

$$\text{so } J_n J_n^T = \begin{bmatrix} (1/9)^{2n} & 0 \\ 0 & 2^{2n} \end{bmatrix} \qquad r_1^n = 2^n \qquad r_2^n = (1/9)^n$$

$$L_1 = \lim_{n\to\infty} (2^n)^{1/n} = 2 \qquad L_2 = \lim_{n\to\infty} ((1/9)^n)^{1/n} = 1/9$$

$$h_1 = \ln 2 \qquad h_2 = \ln(1/9)$$

(b) Are most bounded orbits chaotic? Explain your reasoning.

Yes. Because $h_1 > 0$.

(c) Determine what happens to the area by applying the map by looking at the sum of the Lyapunov exponents.

$h_1 + h_2 = \ln(2/9) < 0 \Rightarrow$ area is decreasing.

(d) Assume the map $\boldsymbol{f}$ on $\mathbb{R}^m$ has constant Jacobian $\mathsf{Df}(\boldsymbol{x})$ with with determinant $D$ for each $\boldsymbol{x}$. Prove that the product of all $m$ of the Lyapunov numbers is $D$ (or equivalently, the sum of the Lyapunov exponents is $\ln D$.)

$J = Df(x)$. - constant matrix. $\Rightarrow J_n = J^n$

For the Lypunov exponent we need the eigenvalues of $J_n J_n^T$

ie $r_k^n = \sqrt{\lambda_k(J_n J_n^T)}$

$$h_k = \lim_{n\to\infty} \ln\left[\left(\sqrt{\lambda_k(J_n J_n^T)}\right)^{1/n}\right]$$

$$\Rightarrow \sum_{k=1}^{m} h_k = \lim_{n\to\infty} \ln\left(\prod_{k=1}^{m}\left(\sqrt{\lambda_k(J_n J_n^T)}\right)^{1/n}\right)$$

Now the $\Pi$ of eigenvalues is = to the det by $J_n J_n^T$. (Via Math22)

$$\det(J_n J_n^T) = \det(J_n)\det(J_n^T)$$
$$= \det(J)^n \det(J^T)^n$$

Now eigenvalues of $J = J^T$.

$$\det(J_n J_n^T) = \det(J)^n \det(J)^n$$

$$D = \det(J) = \prod_{k=1}^{m} \mu_k \quad \mu_k \text{ eigenvalues of } J.$$

$$\Rightarrow \sum_{k=1}^{m} h_k = \lim_{n\to\infty} \ln\left[\left(\prod_{j=1}^{m} \mu_k\right) \cdot \left(\prod_{j=1}^{m} \mu_k\right)\right]^{1/2} = \ln D(J)$$

8

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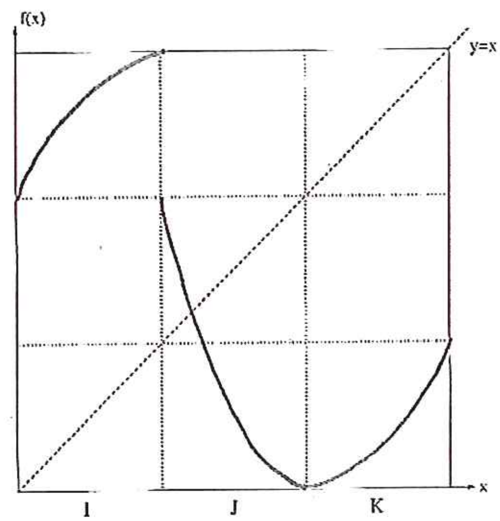
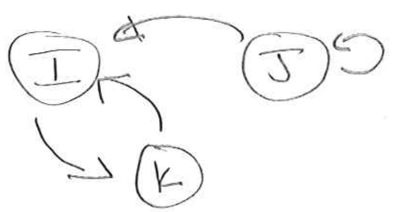
4. Consider the function $f$ with the following graph:




(a) Draw the transition graph.




(b) Give the general statement of the Fixed Point Theorem.

let f be a continuous map of real line. & let I = [a,b]. be an interval st f(I) ⊇ I. Then f has a fixed pt in I.

(c) Are there any periodic orbits starting in $J$? If so, prove their existence.

This was a typo should have read. "Are there any periodic orbits starting in J that are not in J?"

for question statement. Yes: y = x intersects f(x) in J because f(J) ⊇ J, ∴ fixed pt. exist. & is periodic.

For intended question, ∃ periodic orbits of the form $J^n \overline{IK}$, since $J^n$ can map to I & $\overline{IK}$ has 2 periodic orbit.

5. Random short answer questions.

(a) Is $i$ in the Julia set $J(-i)$? If so, is there something special about $i$ under the mapping $P(z) = z^2 - i$?

$i \to -1-i \to (-1-i)^2 - i = (1+2i-1) - i = i$ $\Rightarrow$ 2 periodic

Yes $i \in J(-i)$

(b) Is $-i$ in the Mandelbrot set?

$0 \to -i \to (-i)^2 - i = -1-i \to (-1-i)^2 - i = i.$

Yes

(c) Show that the set of odd positive integers is countable?

You can enumerate the odd integers by

$n_k = 2k+1 \quad k = 1, 2, \ldots$

(d) Is the set $[0, 1]$ countable?

Not countable.

(e) Can there exist a conjugacy relation between a set of measure zero and a set with nonzero measure? If so, give an example including the two sets and the conjugacy relation.

Yes. The middle third cantor set can be mapped to $[0,1]$ via binary #'s.

BONUS Prove that for the map $P_c(z) = z^2 + c$ with $|c| < 2$, if $z_n$ ever leaves the disc of radius 2 about the origin, the iteration will go to infinity. [Be sure to exclude any finite limits. Hint: use the triangle inequality $|a + b| \leq |a| + |b|$ for $a,\ b \in \mathbb{C}$, but you need to get a lower bound on $|z_{n+1}|$].

See Homework 5 solutions