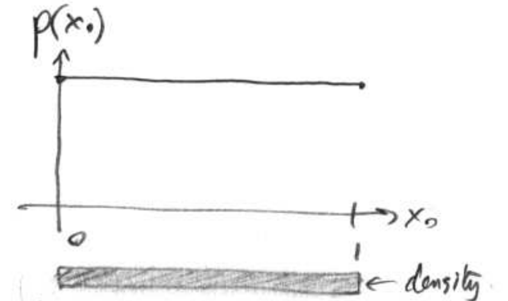
MATH 53 WORKSHEET : Fractals from probabilistic games

10/29/07 Barnett

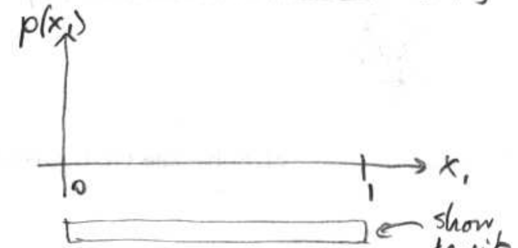
- Apply $f_1(x) = \frac{x}{3}$ or $f_2(x) = \frac{x+2}{3}$ with equal probability of $\frac{1}{2}$ on each iteration.

Starting with $p(x_0)$ uniform in $[0,1]$,

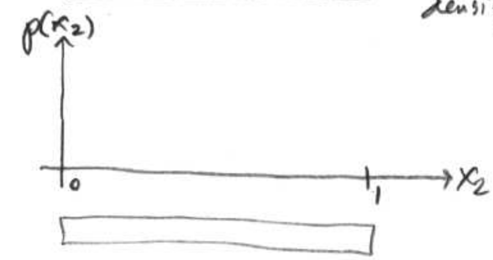


Find $p(x_1)$ and sketch

[Hint: what <u>geometrically</u> does f_2 do?]



Find $p(x_2)$ and sketch

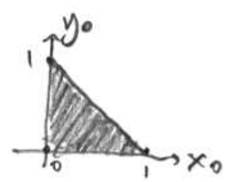


What is $p(x_n)$?

What is the limiting attractor set as $n \to \infty$?

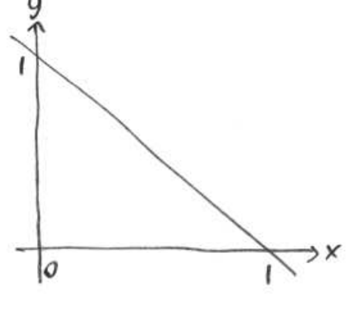
Prove an <u>upper bound</u> on the distance of x_n to this set [Hint: dist of $x_0 \leq$?]

- Now try a 2D example : start with $\bar{x}_0$ uniform in triangle



$\vec{x}_0 = (x_0, y_0)$

Apply
$$\begin{cases} \vec{f}_1(\vec{x}) = \left(\frac{x}{2}, \frac{y}{2}\right) \\ \vec{f}_2(\vec{x}) = \left(\frac{x+1}{2}, \frac{y}{2}\right) \\ \vec{f}_3(\vec{x}) = \left(\frac{x}{2}, \frac{y+1}{2}\right) \end{cases}$$
with equal probabilities $\frac{1}{3}$.



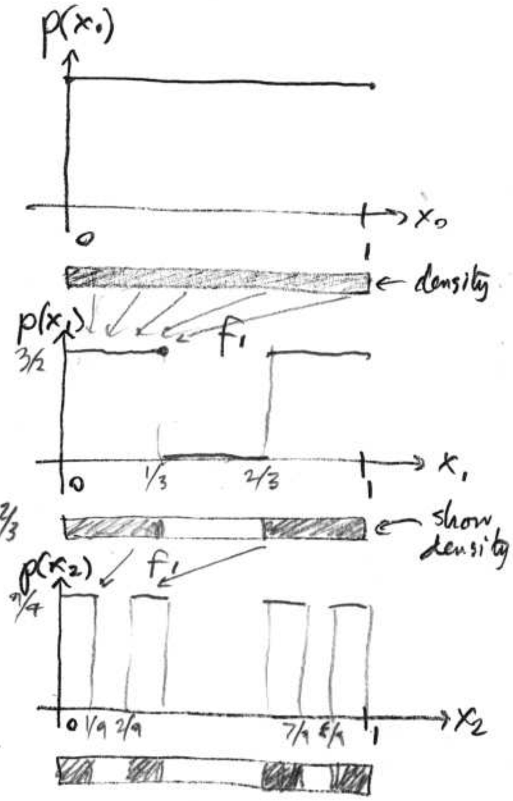
← Deduce the attractor set.

MATH 53 WORKSHEET : Fractals from probabilistic games — SOLUTIONS

10/29/07 Barnett

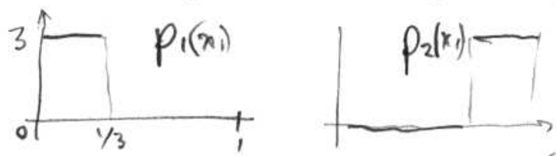
- Apply $f_1(x) = \frac{x}{3}$ or $f_2(x) = \frac{x+2}{3}$ } with equal probability of ½ on each iteration.

Starting with $p(x_0)$ uniform in $[0,1]$,



Find $p(x_1)$ and sketch

[Hint: what geometrically does f_2 do?] average them



$$p(x_1) = \begin{cases} 3/2 & x_1 < 1/3 \text{ or } x_1 > 2/3 \\ 0 & \text{otherwise} \end{cases}$$

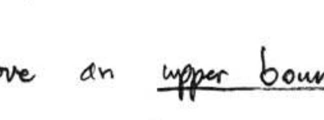
Find $p(x_2)$ and sketch

$$p(x_2) = \begin{cases} 9/4 & x_2 < 1/9 \text{ or } 2/9 < x_2 < 1/3 \text{ or } \\ & 2/3 < x_2 < 7/9 \text{ or } x_2 > 8/9. \\ 0 & \text{otherwise} \end{cases}$$

What is $p(x_n)$? $p(x_n) = \left(\frac{3}{2}\right)^n$ if $x_n \in K_n$, 0 otherwise

What is the limiting attractor set as $n \to \infty$? K_∞, the "missing third" Cantor-Set

Prove an upper bound on the distance of x_n to this set [Hint: dist of $x_0 \leq$?]

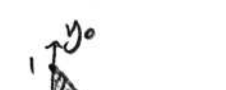


K_∞ worst-case $x_0 = \frac{1}{2}$ has $\text{dist}(x_0, K_\infty) = 1/6$

Upon iteration, gets 3 times closer per iteration

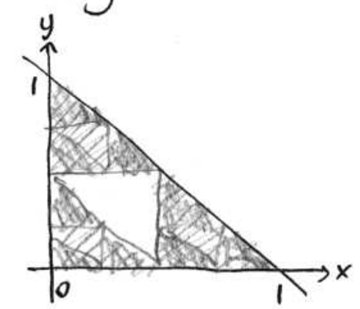
$$\Rightarrow \text{dist}(x_n, K_\infty) \leq \frac{1}{6 \cdot 3^n}$$

- Now try a 2D example : start with $\vec{x}_0$ uniform in triangle



$\vec{x}_0 = (x_0, y_0)$

Apply $\begin{cases} \vec{f}_1(\vec{x}) = \left(\frac{x}{2}, \frac{y}{2}\right) \\ \vec{f}_2(\vec{x}) = \left(\frac{x+1}{2}, \frac{y}{2}\right) \\ \vec{f}_3(\vec{x}) = \left(\frac{x}{2}, \frac{y+1}{2}\right) \end{cases}$ with equal probabilities $\frac{1}{3}$.



← Deduce the attractor set.

geometrically make $\vec{x}$ move half the dist. towards 3 vertices.

Sierpinski gasket for vertices (0,0), (1,0), (0,1).