MATH 53 WORKSHEET: Stable & Unstable manifolds. [T2-9]

10/12/07 Barnett

Consider $\vec{f}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x/2\\2y-7x^2\end{pmatrix}$ map in $\mathbb{R}^2$

A) find formula for $\vec{f}^{-1}\begin{pmatrix}x\\y\end{pmatrix}$

B) Sketch the set $S=\{(x,4x^2): x\in\mathbb{R}\}$

Show $S$ is invariant under $\vec{f}$

ie $\vec{x}\in S \Rightarrow \vec{f}(\vec{x}), \vec{f}^{-1}(\vec{x}) \in S$.

C) Show $S$ is either stable or unstable manifold — which? (where's fixed pt?)

D) What is the other manifold? [Hint: fix $x=0$]

E) Show no points outside $S$ converge to $\vec{0}$ under $\vec{f}$ or $\vec{f}^{-1}$

MATH 53 WORKSHEET: Stable & Unstable manifolds. [T2-9]

10/12/07 Barnett

— SOLUTIONS —

Consider $\vec{f}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x/2\\2y-7x^2\end{pmatrix}$ map in $\mathbb{R}^2$

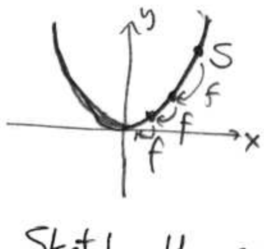
A) find formula for $\vec{f}^{-1}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x\\y/2+14x^2\end{pmatrix}$

Define $\vec{f}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$ ie

solve for $u,v$ $\begin{cases} u/2 = x \\ 2v - 7u^2 = y \end{cases}$

$u = 2x$

$v = \dfrac{y+7u^2}{2} = \dfrac{y+28x^2}{2} = y/2 + 14x^2$




B) Sketch the set $S = \{(x, 4x^2): x \in \mathbb{R}\}$

Show $S$ is invariant under $\vec{f}$

ie $\vec{x} \in S \Rightarrow \vec{f}(\vec{x}), \vec{f}^{-1}(\vec{x}) \in S$.

$\vec{f}\begin{pmatrix}x\\4x^2\end{pmatrix} = \begin{pmatrix}x/2\\2(4x^2)-7x^2\end{pmatrix} = \begin{pmatrix}x/2\\x^2\end{pmatrix} \in S$ since applying "$4x^2$" to $x/2$ gives $x^2$.

$\vec{f}^{-1}\begin{pmatrix}x\\4x^2\end{pmatrix} = \begin{pmatrix}2x\\\frac{4x^2}{2}+14x^2\end{pmatrix} = \begin{pmatrix}2x\\16x^2\end{pmatrix} \in S$.

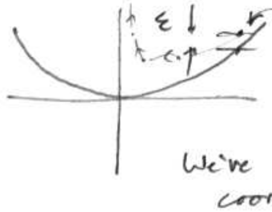
C) Show $S$ is either stable or unstable manifold — which? (where's fixed pt?)

fixed point is $\vec{0}$. since $\begin{pmatrix}x\\4x^2\end{pmatrix} \to \begin{pmatrix}x/2\\x^2\end{pmatrix} \to \begin{pmatrix}x/4\\x^2/4\end{pmatrix} \cdots$ has limit $\begin{pmatrix}0\\0\end{pmatrix}$


D) What is the other manifold? [Hint: fix $x=0$]

$U(\vec{0}) =$ y-axis $\{\begin{pmatrix}0\\y\end{pmatrix}: y \in \mathbb{R}\}$ since $\vec{f}^{-n}\begin{pmatrix}0\\y\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}$ in limit.

E) Show no points outside $S$ converge to $\vec{0}$ under $\vec{f}$ or $\vec{f}^{-1}$




Any point can be written $\begin{pmatrix}x\\4x^2+\varepsilon\end{pmatrix} \xrightarrow{f} \begin{pmatrix}x/2\\x^2+2\varepsilon\end{pmatrix} \to \begin{pmatrix}x/4\\x^2/4+4\varepsilon\end{pmatrix} \cdots \begin{pmatrix}x/2^n\\x^{2n}/4^n\end{pmatrix} + \begin{pmatrix}0\\2^n\varepsilon\end{pmatrix}$

blows up unless $\varepsilon = 0$.

We've done a nonlinear coordinate change $\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x\\\varepsilon\end{pmatrix}$ that makes map linear with $A = \begin{pmatrix}1/2 & 0\\0 & 2\end{pmatrix}$ !