Math 53: Chaos!: Midterm 2, FALL 2011

2 hours, 60 points total, 6 questions worth various points (proportional to blank space)

1. [9 points] Complex dynamics. Please show working or some explanation.

   (a) Is i in the Julia set J(1) ?

   (b) Is 1 in the Mandelbrot set, and why?

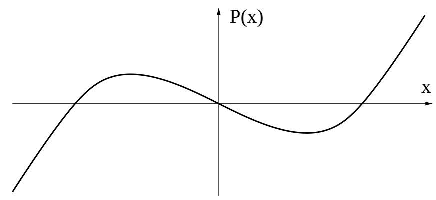
   (c) Consider the map $f(z) := z^2 + 1$, for $z \in \mathbb{C}$. Could there exist a periodic sink for this map?

   (d) Could there exist a $z_0 \in \mathbb{C}$ such that $f^n(z_0)$ remains bounded as $n \to \infty$?

   (e) Based on your answers above, do you expect $J(1)$ to be connected/disconnected? Have nonzero/zero measure? (circle those that apply; no explanation needed)

BONUS: Either find an example of a bounded such z_0 from part (d), or prove there cannot exist any.

2. [16 points] Consider 1D motion of a point particle in the potential $P(x) = x^3/3 - x$, which has roughly the following graph:



(a) Write a system of first-order ODEs for the dynamics in this potential, with no damping.

(b) Sketch the phase plane $(x, \dot{x})$ showing several orbits including all the types of motion that can occur:

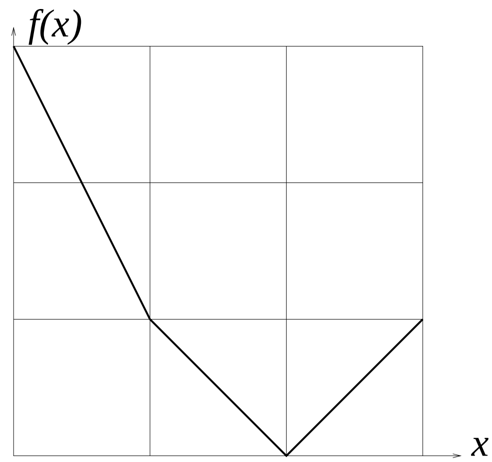
(c) Find all equilibria and categorize their stability. Justify your stabilities by giving a rigorous argument in each case. [Hint: use the phase plane]

(d) In what set of energies do periodic orbits lie? [take care with endpoints]

(e) Sketch the set of all phase plane points which have the unstable equilibrium as their limit as $t \to \infty$.

(f) Imagine a small amount of damping is now added. Sketch on a phase plane the basin of the stable equilibrium.

3. [10 points] Consider the continuous function f with the following graph:



(a) Draw the transition graph (use three intervals A, B, and C):

(b) Prove that a period-2 orbit exists.

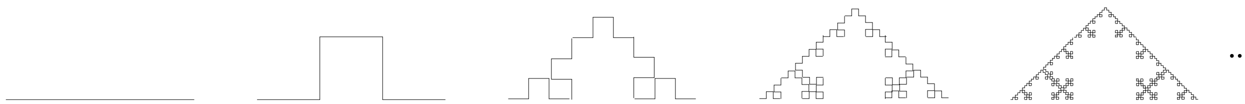
(c) Can a period-3 orbit exist? Prove your answer.

(d) List all periods that *must* exist, giving a proof of your answer. [Hint: check the obvious before you get fancy]

BONUS: Assuming that f is linear in each of the three intervals, prove that if a period-4 orbit exists, it must enter all three intervals.

4. [6 points]

(a) Find the box-counting dimension of the curve (a subset of $\mathbb{R}^2$) formed by the limiting process sketched below: for each straight line segment remove the middle third and replace it by the other three sides of the square. [Hint: describe your 'boxes'. To avoid colliding boxes you may rotate them to cover without collisions]



(b) Could there be a subset of $\mathbb{R}$ with this same box-counting dimension? Explain.

BONUS: Describe an alternative construction whose limit gives this same fractal

5. [8 points] Consider the following map acting on points (x, y) in the unit square $[0, 1]^2$:

$$B(x, y) = \begin{cases} (\frac{x}{4},\ 2y(\text{mod } 1)), & \text{if } y < 1/2, \\ (\frac{x+3}{4}, 2y(\text{mod } 1), & \text{otherwise} \end{cases}$$

(a) What is the complete set of Lyapunov exponents (for almost all initial conditions) for this map?

(b) Orbits of this map are attracted to a limiting set in $\mathbb{R}^2$. Is (4/5, 1/4) in this set, and why?

(c) What is the box-counting dimension of this attractor in $\mathbb{R}^2$?

(d) Let $(x_0, y_0) \to (x_1, y_1) \to \dots$ be any orbit with (x_0, y_0) in the unit square. Give a tight upper bound on the distance of (x_n, y_n) from the attractor.

6. [11 points] Random short-answer questions

(a) What can you deduce about the ODE system $\dot{\mathbf{x}} = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^4$ given that the matrix A has eigenvalues -3, -1, 0, and 0?

(b) Give the mathematical definition of an equilibrium point $\mathbf{v}$ of a flow being *stable*.

(c) What is the measure of the set of points in $[0,1]$ whose *decimal* expansion never uses the digit "0" ?

Prove if this set is finite, countably infinite, or uncountably infinite. (You may use known properties of the set $[0,1]$.)

(d) What is the Lyapunov exponent of almost all bounded orbits of $G(x) = 4x(1-x)$? Explain why.

(e) Prove that there exists an orbit of $G(x) = 4x(1-x)$ that is dense in [0,1].