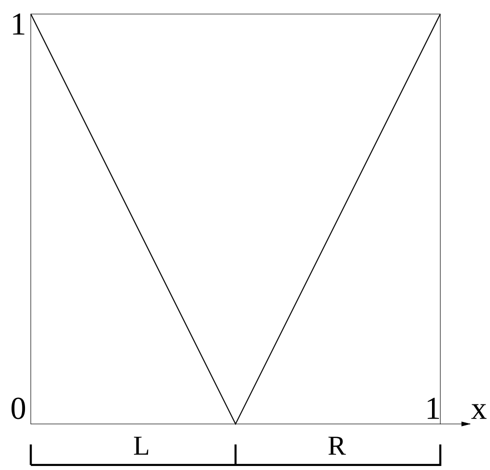
Math 53: Chaos! 2011: Midterm 1

2 hours, 56 points total, 5 questions worth various points ($\propto$ blank space). Good luck!

1. [16 points] Consider the 1D map given by $f(x) = |2x - 1|$, as shown here, on $[0, 1]$, which has been labelled with intervals L and R.



   (a) Write here the subintervals down to level 3 (that is, the correct ordering of all 3-symbol itinerary subintervals on $[0, 1]$):

   (b) Compute the Lyapunov exponent of almost all orbits of this map.

BONUS: Describe the binary representation of all initial points in $[0,1]$ which do *not* have this exponent

(c) Imagine that a computer running a standard numerical environment (such as MATLAB) is used to iterate f starting at x_0 the only fixed point in $(0,1)$. Describe what will happen, including an estimate of how long it will take for errors to become of size $O(1)$. [Hint: numbers are represented with relative error of order 10^{-16}.]

(d) It is not hard to see that f^k has 2^k fixed points for each natural number k. Compute the number of *periodic orbits* of period 4.

(e) Sketch a proof that each point x_0 in $[0,1]$ has sensitive dependence on initial conditions.

2. [11 points] Consider the two-dimensional map $f(\mathbf{x}) = A\mathbf{x}$.

(a) Consider the case $A = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$. What is the closest distance to the origin that a point $\mathbf{x}$ lying on the unit circle $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$ can get mapped to ? [Hint: easier if bring out the common factor in the matrix]

(b) Your answer should be consistent with some points on the unit circle moving closer to the origin. For this A, find all possible fate(s) of orbits starting on the unit circle (ie, to where they may tend upon repeated iteration).

(c) What is the area enclosed by the image of the unit circle under one application of the above map?

(d) Now let $A = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 3/2 \end{bmatrix}$. Is the origin now a *hyperbolic* fixed point?

(e) Keeping A as in (d), either find the *complete* set of points which tend to the origin upon repeated iteration, or explain why this set is empty.

3. [9 points]

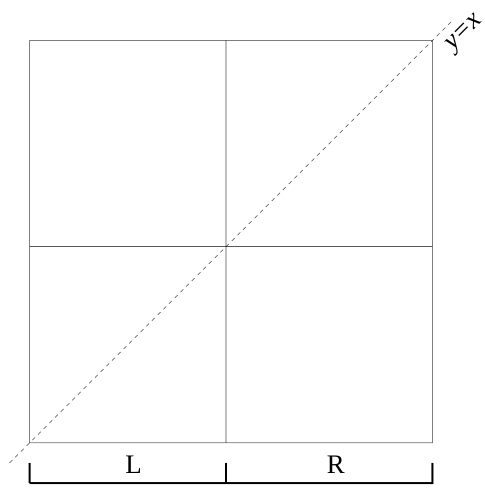
Consider the Henón map on $\mathbb{R}^2$ with $b = 1/2$ and general a, that is, $f(x, y) = (a - x^2 + y/2, x)$.

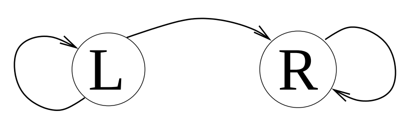
(a) Find the fixed point(s) in terms of a. For what a does at least one fixed point exist?

(b) Now specialize to $a = 7/4$. What is the period of the orbit from point $(3/2, -1)$?

(c) Categorize the stability of the orbit from (b). Is it a (possibly periodic) sink, saddle, source, or none of these?

4. [8 points]





(a) Using the axes above, with the partition L and R as shown, draw a possible graph of a smooth function f mapping $L \cup R$ into $L \cup R$ with transition graph as shown to the right.

(b) Onto what interval does the subinterval LR get mapped by f?

(c) Give a list of *all* the types of itineraries that *must* occur given the transition graph:

(d) Given only the above information about f, is it possible that there exists a period-2 orbit? Explain.

5. [12 points] Random short questions. Please explain each briefly.

(a) The origin is a fixed point of $f(x) = x + x^2$. Categorize it as a source, sink, or neither.

(b) What is the Lyapunov exponent of almost all bounded orbits of $f(x) = 0.9x(1 - x)$ on $\mathbb{R}$? [Hint: sketch]

(c) True/False: the basin of a sink for any smooth map $f : \mathbb{R} \to \mathbb{R}$ must consist of an interval (possibly unbounded)?

(d) Let p be a fixed point of a map on $\mathbb{R}^m$. Give the mathematical definition of p being a source.

(e) Compute (by hand!) the binary representation of the fraction 7/9

(f) Consider a 1-dimensional map undergoing bifurcation with respect to some parameter. What is the Lyapunov *number* of the orbit at its bifurcation point?