MATH 53 : Points in Mandelbrot set

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10/29/09

Defn: $\mathcal{M} = \{ c \in \mathbb{C} :$ 0 is not in basin of ∞ for map $P_c(z) := z^2 + c \}$

a) is $c = -1$ in $\mathcal{M}$?

b) is $c = 1$ in $\mathcal{M}$?

How far did you have to go to believe this?

c) is $c = i$ in $\mathcal{M}$?

d) check the stability of the orbit you just found — sink, source, saddle?
[Hint either use $\binom{x}{y}$ map in $\mathbb{R}^2$ or cheat & use 1d formula!]

What does Fatou theorem tell you about if another sink could exist?

So what shape/size is $J(c)$ for $c = i$?

Do you expect $c = i$ to be in interior/boundary/exterior of $\mathcal{M}$? [Hint: perturb c]

e) find a simple linear conjugacy between $P_c(z) = z^2 + c$, c, z real, and $g_a(x) = ax(1-x)$, for some a related to c.

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SOLUTIONS

Defn: $\mathcal{M} = \{ c \in \mathbb{C} : 0$ is not in basin of ∞ for map $P_c(z) := z^2 + c \}$

a) is $c = -1$ in $\mathcal{M}$? iterate: $0 \to -1 \to \underbrace{(-1)^2 - 1 = 0 \to -1}_{\text{period-2}} \to$ etc

not going to $\infty \Rightarrow$ yes.

b) is $c = 1$ in $\mathcal{M}$? $0 \to 1 \to 1^2+1 = 2 \to 5 \to 26 \to \dots \infty \Rightarrow$ no.

How far did you have to go to believe this? exceeds 2 so its fate is sealed: it goes to ∞.

c) is $c = i$ in $\mathcal{M}$? $0 \to i \to i^2 + i = -1+i \to (i-1)^2 + i = -i$

period-2, bounded $\Rightarrow$ yes.

d) check the stability of the orbit you just found — sink, source, saddle

[Hint either use $\binom{x}{y}$ map in $\mathbb{R}^2$ or cheat & use 1d formula!]

$f\binom{x}{y} = \begin{pmatrix} x^2 - y^2 + a \\ 2xy + b \end{pmatrix}$ where $c = a + ib$, $c = i$

$= \begin{pmatrix} x^2 - y^2 \\ 2xy + 1 \end{pmatrix}$ so $Df = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$

$Df^2\binom{0}{-1} = Df\binom{-1}{1} Df\binom{0}{-1} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & +2 \\ +2 & 0 \end{pmatrix}$

$= \begin{pmatrix} -4 & 4 \\ -4 & -4 \end{pmatrix}$ eigvals $4 \pm 4i$ magnitude $4\sqrt{2} > 1$

$\Rightarrow$ unstable, source.

What does Fatou theorem tell you about if (another) sink could exist?

Sinks must have 0 in their basin, so no sink exists.

So what shape/size is $J(c)$ for $c = i$? it must have zero measure (but be connected, since $i \in \mathcal{M}$).

Do you expect $c = i$ to be in interior/boundary/exterior of $\mathcal{M}$? [Hint: perturb c]

since slight perturbation leads to falling off the unstable period-2 & going to ∞.

e) find a simple linear conjugacy between $P_c(z) = z^2 + c$, c, z real, and [Harder!] $g_a(x) = ax(1-x)$, for some a related to c.

$x = \frac{1}{2} + \frac{z}{a}$ $\quad c = \frac{a}{2} - \frac{a^2}{4}$ (see books for this ...)