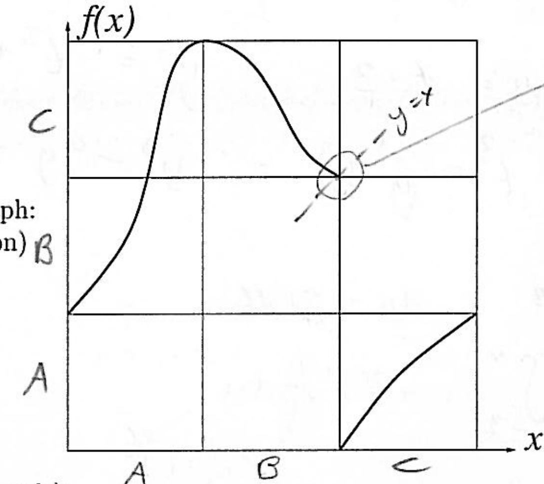
~ SOLUTIONS ~

Math 53: Chaos!: Midterm 2, FALL 2009

2 hours, 60 points total, 5 questions worth various points (proportional to blank space)

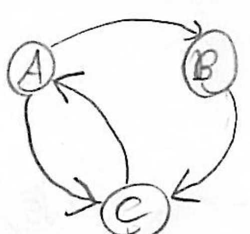
1. [10 points]

Consider the function f with the following graph:
(You may assume f is monotonic in each region)



Some of you realised that if the endpoint was included here, there would be a period-1 orbit at the junction of B & C.
I intended: (which rules this out) ✓

3 (a) Draw the transition graph (use three intervals):



no repetition allowed (but see above).

meaning, and give a...

note: fixed pts of f^6 are ABCABCA (could be p-3) & ACACACA (could be p-2)

4 (b) Which of the following periods can you prove must exist? (give a proof for just *one* of these cases):

1   2   3   5   6

ABCA

ABCACA

note cannot be factorized into lower periods.

Proof:

since $f^2(ACA) = A \supset ACA$ (symbol shift)

then by the fixed pt. theorem there exists a fixed pt of f^2 in ACA subinterval.

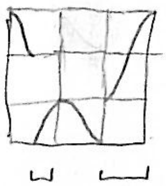
This cannot be due to a period-1 since it moves from A to C.

$\Rightarrow$ Period-2 exists.

3 (c) Prove that a period-4 orbit *cannot* exist. [Hint: consider monotonicity of f^2 in some subinterval]

f^2 is monotonically increasing in the subintervals AC and C

f^2:



Suppose a period-4 exist, then this is a fixed pt of f^4, so must have itinerary $\overline{ACAC}$, and $f^2(p_1) = p_2$, $f^2(p_2) = p_1$ for points $p_1 \neq p_2$ in AC. Say: $p_2 > p_1$ then $f^2(p_2) > f^2(p_1)$ by monotonicity, but this says $p_1 > p_2$, a contradiction. The same applies, if $p_2 < p_1$, QED.

[BONUS: what periods above 6 must exist and why?]

All odd k periods exist since $ACAC\cdots ACABC$ valid & not factorizable into a divisor of k.
$(k \geq 7)$

All even k periods exist since $AC\cdots ACABCABC$, similarly.
$(k \geq 8)$

So all periods $k > 6$ exist!

2. [9 points] Consider, on the unit square, the linear torus map $T(\mathbf{x}) = A\mathbf{x}$ (mod 1), where A is a 2×2 matrix with integer entries.

4 (a) In the case $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, compute all Lyapunov exponent(s) of the map. see lecture; $A = P^{-1}\Lambda P$

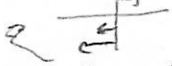
$$\underset{A=A^T}{\text{symm}} \Rightarrow \text{diagonalizable} \Rightarrow \lambda(A^n A^{nT}) = \lambda(A^{2n}) = \lambda(A)^{2n}$$

eigvals of A: $(1-\lambda)(2-\lambda) - (-1)^2 = 0 \rightarrow \lambda^2 - 3\lambda + 1 = 0$

$$\lambda_{1,2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{9-4} = \frac{3 \pm \sqrt{5}}{2}$$

ellipsoid axes $r_j^{(n)} = \sqrt{\lambda_j(A^n A^{nT})} = \lambda_j(A)^n \qquad j = 1,2$

Lyapunov exponents $h_j = \lim_{n\to\infty} \frac{1}{n} \ln r_j^{(n)} = \frac{n}{n} \ln \lambda_j(A) = \begin{cases} \ln \frac{3+\sqrt{5}}{2} \\ \ln \frac{3-\sqrt{5}}{2} \end{cases}$



shear map

none of these! (tricky)

3 (b) Now for the case $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, is $\mathbf{0}$ a source/sink/saddle? [Hint: consider the action of A^n on the point $(0, \varepsilon)$ for arbitrarily small ε]. Compute the Lyapunov exponent(s) and explain the discrepancy between this and whether $\mathbf{0}$ is a sensitive point.

$$A\begin{bmatrix}0\\ \varepsilon\end{bmatrix} = \begin{bmatrix}\varepsilon\\ \varepsilon\end{bmatrix}, \quad A\begin{bmatrix}n\varepsilon\\ \varepsilon\end{bmatrix} = \begin{bmatrix}(n+1)\varepsilon\\ \varepsilon\end{bmatrix} \quad \underset{\text{(induction)}}{\text{so}} \quad A^n\begin{bmatrix}0\\ \varepsilon\end{bmatrix} = \begin{bmatrix}n\varepsilon\\ \varepsilon\end{bmatrix}$$

Thus there are points arb. close to $\vec{0}$ (as $\varepsilon \to 0$) that eventually maps far away.
($\Rightarrow$ sensitive dependence).

A has eigenvalues 1 (twice), so not hyperbolic, not a source
(not all points in neighborhood other than $\vec{0}$ leave the neighborhood).

$$r_j^{(n)} = \sqrt{\lambda_j(A^n A^{nT})} = \lambda_j\left(\begin{bmatrix}1&n\\0&1\end{bmatrix}\begin{bmatrix}1&0\\n&1\end{bmatrix}\right)^{1/2} = \lambda_j^{1/2}\left(\begin{bmatrix}1+n^2&n\\n&1\end{bmatrix}\right)$$ look at its growth

eigvals are $\lambda^2 - (2+n^2)\lambda + 1 + \cancel{n^2} - \cancel{n^2} = 0$ so $\lambda = 1 + \frac{n^2}{2} \pm \sqrt{\left(\frac{2+n^2}{2}\right)^2 - 1} = O(n^2)$

so $h_j = \lim_{n\to\infty} \frac{1}{n} \ln \lambda_j^{1/2}\left(\begin{bmatrix}1+n^2&n\\n&1\end{bmatrix}\right) = \lim_{n\to\infty} \frac{1}{n} \ln$ (something growing at most like n^2)

$= \lim_{n\to\infty} \frac{c}{n} \ln n = \underline{0}$, $j=1,2$. Sens. dep. is very weak since growth is linear not exponential. (hard)

2 (c) Prove that, for general A, if the map T is area-preserving (hence invertible) then the Lyapunov exponents of T and T^{-1} are the same.

T area-preserving $\Leftrightarrow$ $\det A = 1$

But $\sum_{j=1}^{2} h_j = \ln \det A = 0$ So $h_2 = -h_1$ (check true for (a))

If v eigvec of A w/ eigval λ, $v = A^{-1}Av = A^{-1}\lambda v = \lambda A^{-1} v$

so $A^{-1}v = \lambda^{-1}v$, and A^{-1} has same eigvectors but inverse eigvals.

So T^{-1} has negatives of Lyap. exps. of T, but since $h_2 = -h_1$ these stay the same.

3. [14 points] Consider 1D motion of a point particle in the potential $P(x) = x^2 - x^4$.

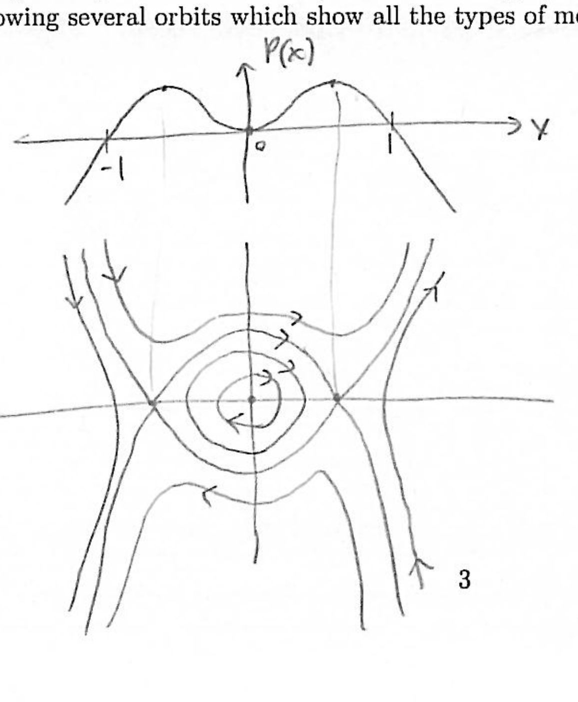
1 (a) Write a system of first-order ODEs for the dynamics in this potential, with no damping.

2nd order: $\ddot{x} = \text{force} = -\frac{dP}{dx} = -2x + 4x^3$

1st order: $\dot{x} = y$

$\dot{y} = -2x + 4x^3$

3 (b) Graph the potential function (careful about signs) and below that, graph the phase plane $(x, \dot{x})$ showing several orbits which show all the types of motion that can occur:



x^2 dominates for $|x| \ll 1$

$-x^4$ " " $|x| \gg 1$

5 (c) Find all equilibria and categorize their stability. Justify your stabilities by giving rigorous arguments. [Hint: use the phase plane]

equilibria in x where force $= -\frac{dP}{dx} = 0 \Rightarrow -2x + 4x^3 = 0$

$\Rightarrow x\left(\frac{1}{2} - x^2\right) = 0$

$\Rightarrow x = 0, \pm\frac{1}{\sqrt{2}}$

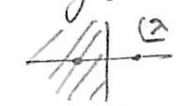
linearize

$\boxed{x=0}$: $A = Df\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -2+12x^2 & 0 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \xrightarrow{\text{eigvals}} \lambda^2 + 2 = 0, \quad \lambda = \pm\sqrt{2}\,i$

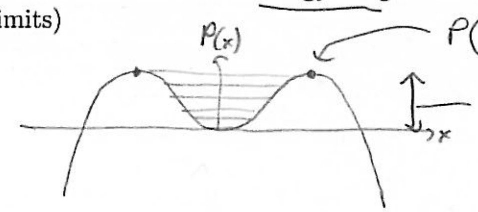
cannot deduce stability from linearization thm.

However, since $E(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + P(x)$ conserved, in any neighborhood N of $\vec{0}$ there is a contour line of E (>0) in which a subset $N_1 \subset N$ is trapped inside N. $\Rightarrow$ (Lyapunov) Stable.

$\boxed{x = \pm\frac{1}{\sqrt{2}}}$: $Df\Big|_{(\frac{1}{\sqrt{2}},0)} = \begin{bmatrix} 0 & 1 \\ -2+\frac{12}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \xrightarrow{\text{eigvals}} \lambda = \pm 2$ so by stability thm, it's a saddle



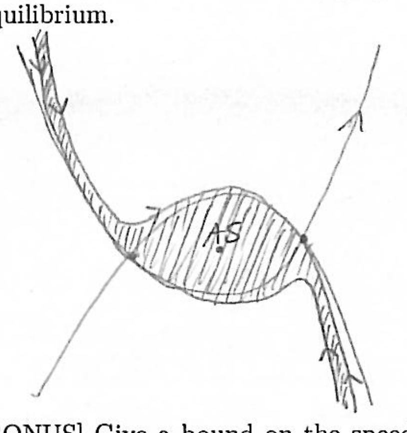
2 (d) What is the allowable energy range where bounded motion can happen? (give upper and lower limits)



$P\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{2^2} = \frac{1}{4}$

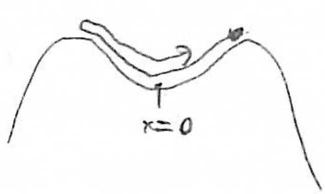
bounded orbits for $0 \leq E \leq \frac{1}{4}$

2 (e) Imagine a small amount of damping is now added. Sketch on a phase plane the basin of the stable equilibrium.



basin is bounded by the stable manifolds of the saddle points.

[BONUS] Give a bound on the speed of a particle which passes through the stable equilibrium more than once.



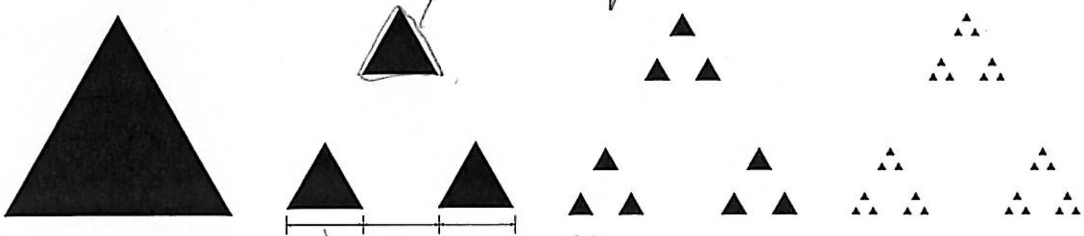
for $0 < E < 1/4$, particle passes $x=0$ repeatedly (periodically).

$P(0) = 0$ so all of E is accounted for by $\frac{1}{2}\dot{x}^2 < 1/4$ so $|\dot{x}| < \frac{1}{\sqrt{2}}$

4. [13 points]

4 (a) Find the box-counting dimension of the 'triangular Sierpinski carpet' set given by the limit of the process shown applied to the equilateral triangle: (in each step the three lengths as shown are equal)

use triangles as boxes.



$\varepsilon = 1$ $\quad$ $\varepsilon = 1/3$ $\quad$ $\varepsilon = 1/3^2$ $\quad \ldots \quad$ $\varepsilon = 1/3^n$

$N(\varepsilon) = 1$ $\quad$ $N(\varepsilon) = 3$ $\quad$ $N(\varepsilon) = 3^2$ $\quad \ldots \quad$ $N(\varepsilon) = 3^n$

$$\text{boxdim } d = \lim_{n\to\infty} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon} = \lim_{n\to\infty} \frac{\cancel{n} \ln 3}{\cancel{n} \ln 3} = 1.$$

(strange for a 2dim set).

2 (b) Describe a probabilistic game whose attractor is the above fractal. (You may use words rather than equations, but be clear and concise.)

iterated function system: with probability $1/3$, move $2/3$ of the distance towards one of the 3 vertices of equilateral triangle.

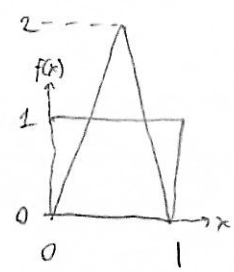
Ie $\quad f_i(\vec{x}) = \frac{1}{3}\vec{x} + \frac{2}{3}\vec{v}_i$, $\quad i = 1 \ldots 3$, $\quad \vec{v}_i$ = i^{th} vertex of triangle

$\vec{v}_1 = (0,0)$, $\vec{v}_2 = (1,0)$, $\vec{v}_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

4 (c) Find the box-counting dimension of the set of initial values whose orbits remain bounded for all time, under the one-dimensional map

$$f(x) = \begin{cases} 4x, & x \le 1/2 \\ 4(1-x), & x > 1/2 \end{cases}$$

[Hint: graph f. Partial credit given for describing the type of set.]



$0 \;\; 1/4 \quad 3/4 \;\; 1.$ $\quad K_1$

After 1 iteration of f, points remaining in $[0,1]$ are K_1

" $\;$ n $\;$ " $\quad$ " $\quad$ " $\;$ K_n.

Middle-½'s Cantor set: K_1 — — ε $N(\varepsilon)$

		ε	$N(\varepsilon)$
K_1	— —	$1/4$	2
K_2	- - - -	$1/4^2$	2^2
K_3	.. ..	$1/4^n$	2^n
$\vdots$			

$$d = \lim_{n\to\infty} \frac{\ln(2^n)}{\ln(4^n)} = \lim_{n\to\infty} \frac{n \ln 2}{n \ln(2^2)} = 1/2 .$$

1 (d) Does the set in (c) contain a finite, countably infinite, or an uncountably infinite number of points? [BONUS: prove your answer]

proof: points in K_∞ have quaternary expansion which can be written as $0\cdot3300303330\cdots$ ie only 0 & 3 used.

We have a 1-1 map onto the binary numbers in $[0,1]$ by replacing "3" by "1". The set $[0,1]$ is uncountably infinite (Cantor's diagonal proof).

2 (e) Give an example of a sequence of box sizes ε tending to zero that would *not* be appropriate for computing box-counting dimension.

You need for box dim $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$ to be valid.

Choose for invalid seq. $\frac{\ln b_{n+1}}{\ln b_n} = 2 \neq 1$ ie $\ln b_{n+1} = 2\ln b_n = \ln(b_n^2)$

So $b_1 = \frac{1}{2}$, $b_{n+1} = b_n^2$, $n=1,2,\cdots$ is a seq, ie $b_n = 2^{-2^n}$ Or, eg. $10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}, \ldots$ (not 2^{n^2}).

5. [14 points] Random short-answer questions

3 (a) Among a set of 10^4 points there are 10^5 pairs of points lying within Euclidean distance 0.1 of each other, but only 10^2 pairs lying within distance 0.001 of each other. Use this to estimate the correlation dimension of the set.

$C(r) = k r^d$ if corr dim d is well-defined.

r	$C(r)$
10^{-1}	10^5
10^{-3}	10^2

Given two data points, $\frac{C_1}{C_2} = \left(\frac{r_1}{r_2}\right)^d$ so $d = \frac{\ln(C_1/C_2)}{\ln(r_1/r_2)}$

$$= \frac{\ln 10^3}{\ln 10^2} = 3/2$$

4 (b) Consider the maps $f(x) = 4x(1-x)$ and $g(x) = 2 - x^2$. Prove that they are conjugate under the (linear) bijection $y = C(x) = 4x - 2$. If the Lyapunov exponent of f is $\ln 2$, what can you deduce (if anything) about the Lyapunov exponent of g?

$$\begin{tikzcd} x_0 \arrow[r, "f"] \arrow[d, "C"'] & x_1 \arrow[d, "C"] \\ y_0 \arrow[r, "g"] & y_1 \end{tikzcd}$$

'commutative diagram'

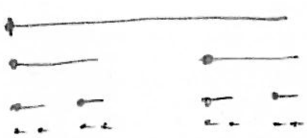
As functions, need $C(f(x)) = g(C(x))$ $\forall x$

$4(4x(1-x)) - 2 = 16x - 16x^2 - 2$

And $g(C(x)) = 2 - (4x-2)^2 = 16x - 16x^2 - 2$ } equal ✓.

Conjugacy implies same Lyapunov exponent, so g also has exponent $\ln 2$.

3 (c) Characterize the set of all left-endpoints remaining in the middle-thirds Cantor set using the ternary system. Is this set countably or uncountably infinite?



$\cdot 0$

$\cdot 0, \cdot 2$

$\cdot 00, \cdot 02, \cdot 20, \cdot 22$

$\cdot 000, \cdot 002$, etc.

left-endpoints have terminating ternary expansion made only of 0's & 2's.

Since the set can be enumerated in order (length-1 strings, then length-2 strings, etc.) it is countable (countable union of finite sets).

1 (d) Is the middle-thirds Cantor set dense in [0,1]?

K^∞

$[0,1]$

K^∞ is not dense in $[0,1]$ since the point $x = 1/2$ is a distance at least $1/6$ from all points in K^∞.

2 (e) Is the point -2 in the Mandelbrot set?

$c = -2$, evolve $z_{n+1} = z_n^2 + c$ from $z_0 = 0$

$0 \to 0^2 - 2 = -2 \to (-2)^2 - 2 = 2 \to 2^2 - 2 = 2 \to 2 \to 2 \to \dots$ eventually periodic

$\Rightarrow$ yes, in Mandelbrot set! (just)

1 (f) Is the point i in the Julia set for $c = -1$?

$z_0 = i \to i^2 - 1 = -2 \to (-2)^2 - 1 = 3 \to 3^2 - 1 = 8 \to \dots \infty$.

$\uparrow$ has left radius 2 (since $|c| < 2$).

no, not in Julia set.