Alex Barnett
10/21/09.

· ~~ SOLUTIONS ~~ · ·

Math 53: Chaos! 2009: Midterm 1

2 hours, 54 points total, 6 questions worth various points (proportional to blank space)

1. [9 points] Consider the two-dimensional map $\mathbf{x} \to A\mathbf{x}$. ← it's a linear map

3 (a) If $A = \begin{bmatrix} 1 & -1/2 \\ 1/2 & 0 \end{bmatrix}$, describe the object formed by applying the map to the unit disc $\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < 1\}$. Include all relevant lengths and directions (unnormalized direction vectors are fine).

$$AA^T = \begin{pmatrix} 1 & -1/2 \\ 1/2 & 0 \end{pmatrix}\begin{bmatrix} 1 & 1/2 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 5/4 & 1/2 \\ 1/2 & 1/4 \end{bmatrix}$$

find eigenvalues λ:

$$\lambda^2 - 3/2\,\lambda + \underbrace{\frac{5}{16} - \frac{1}{4}}_{1/16} = 0$$

quadratic eqn $\lambda = \frac{1}{2}\left(3/2 \pm \sqrt{\frac{9}{4} - \frac{1}{4}}\right) = \frac{3}{4} \pm \frac{\sqrt{2}}{2}$

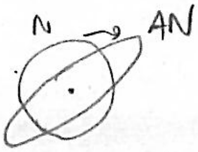
$\lambda_1 = \frac{3}{4} + \frac{\sqrt{2}}{2}$: $\begin{bmatrix} 5/4 - 3/4 - \frac{\sqrt{2}}{2} & 1/2 \\ 1/2 & -1/2 - \frac{\sqrt{2}}{2} \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\frac{1-\sqrt{2}}{2} v_1 + \frac{v_2}{2} = 0$. a bit annoying.

$v_2 = (\sqrt{2} - 1) v_1$

$\vec{v} = \begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix}$. (I didn't grade wrong eigvecs).

direction of semimajor axis.

1 (b) For this A, do any points in the unit disc get mapped outside the unit disc?



yes, since one eigenvalue $\lambda_1 = \frac{3}{4} + \frac{\sqrt{2}}{2}$
$\approx 0.75 + 0.71 > 1$

3 (c) For this A, find the fixed point(s) of the map and classify them.

eigvals. of A itself are via $\begin{vmatrix} 1-\lambda & -1/2 \\ 1/2 & -\lambda \end{vmatrix} = 0 = \lambda^2 - \lambda + 1/4$.

so $\lambda = \frac{1 \pm \sqrt{1-1}}{2} = 1/2$ (twice)

Both λ's are < 1 in magnitude $\Rightarrow$ $\vec{0}$ is a sink

(the only fixed point).

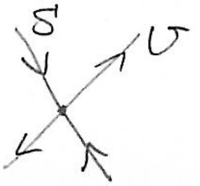
[surprising since the ellipse looks like it's starting to stretch ... eventually it collapses back inside N].

2 (d) Now if $A = \begin{bmatrix} 9/2 & -4 \\ 2 & -3/2 \end{bmatrix}$, does the map have any points with *sensitive dependence*? If so, give a proof for one such point. If not, explain why and categorize any fixed point(s). [Partial credit given for correct definition of sensitive dependence].

eigvals of A: $\lambda^2 - \overbrace{(9/2 - 3/2)}^{3}\lambda + \underbrace{\frac{-9(3)}{4} + 8}_{5/4} = 0$ $\quad \lambda^2 - 3\lambda + 5/4 = 0$

$\Rightarrow \lambda = \frac{3 \pm \sqrt{9-5}}{2} = 1/2,\ 5/2.$

<1 in size, >1 in size → saddle ⇒ yes.



Since $\vec{0}$ is a saddle fixed point, we may choose points $\vec{x} = \varepsilon \vec{v}$, where $\vec{v}$ is the eigenvector with $\lambda_2 = 5/2$, and, no matter how small $\varepsilon > 0$ is, $A^k \vec{x} = \lambda_2^k \varepsilon \vec{v}$ will eventually leave any fixed neighborhood of the point $\vec{0}$. □

[in fact since map is linear, all points have sens. dep.]

2. [10 points] Consider the two-dimensional map $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ a-y^2 \end{pmatrix}$

3 (a) Solve for all fixed points of f. For what range of a do (real) fixed points exist?

Fixed: $\vec{f}(\vec{x}) = \vec{x}$ ie $\begin{aligned} 2x+y &= x \\ a-y^2 &= y \end{aligned}$ $\Rightarrow$ $\begin{aligned} x+y &= 0 \text{ or } y=-x \\ y^2+y-a &= 0 \end{aligned}$ ⮐ solve

$y = \frac{-1 \pm \sqrt{1+4a}}{2}$

so for $a > -1/4$, square root is real, and $\left(+\frac{1+\sqrt{1+4a}}{2}, \frac{-1-\sqrt{1+4a}}{2}\right)$

and $\left(+\frac{1-\sqrt{1+4a}}{2}, \frac{-1+\sqrt{1+4a}}{2}\right)$

are the two fixed points.

validate your finding in ⓐ, at least for a=0!

(b) Fix $a = 0$, and for each of the two fixed points, answer: is it hyperbolic? Can you deduce if it is a sink, source, or saddle? [Hint: first find the y values].

1. FIXED POINT 1: say $(0, 0)$ (by sub. $a=0$ in above).

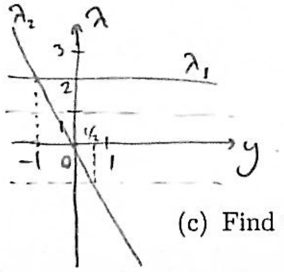
$\vec{Df}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -2y \end{pmatrix}$ so $y=0$ gives $\vec{Df}(\vec{0}) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$

eigvals (since upper-triangular) are $\lambda = 0, 2$ ⇒ saddle, hyperbolic (since $|\lambda_j| \neq 1 \ \forall j$)

2 FIXED POINT 2: $(+1, -1)$ $y=-1$ so $\vec{D}\vec{f}\begin{pmatrix}+1\\-1\end{pmatrix} = \begin{pmatrix}2 & 1\\ 0 & +2\end{pmatrix}$

eigvals are $\lambda = 2$ (twice), both $|\lambda_j| > 1$ $\forall j$ so source

again hyperbolic $(|\lambda_j| \neq 1, \forall j)$



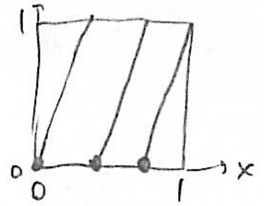
(c) Find the critical value of a above which both fixed points are of the same type.

For general y of fixed point, $\vec{D}\vec{f}$ has eigenvalues $\lambda = \overset{\lambda_1}{2}, \overset{\lambda_2}{-2y}$

so any fixed pts. are sources or saddles. When $|\lambda_2| > 1$ then a fixed point is a source. Looking at plot, this is the large-y (hence, large-a) case. $|\lambda_2| = 1$ when $y = +1/2$, ie $\frac{1}{2} = \frac{-1+\sqrt{1+4a}}{2}$ ie $2^2 = 1+4a$ ie $a = 3/4$. So, for $a > 3/4$, both are sinks. [Tricky].

3. [10 points] Consider the $f(x) = 3x \pmod 1$ which maps the interval $[0, 1)$ to itself.

2 (a) $x_0 = \frac{3}{26}$ is a fixed point of period k. Find k



$$\frac{3}{26} \xrightarrow{f} \frac{9}{26} \xrightarrow{f} \underset{(\text{mod } 1)}{\frac{27}{26} = \frac{1}{26}} \xrightarrow{f} \frac{3}{26}$$

so the smallest k for which $f^k(x_0) = x_0$ is $k=3$.

$\Rightarrow$ this is the period

2 (b) Is this a periodic sink, periodic source, or neither? (show your calculation)

Stability of periodic orbit given by $|f'(p_1)f'(p_2)f'(p_3)| = |(f^3)'(p_1)|$

but $f'(x) = 3$ $\forall x \neq \frac{1}{3}, \frac{2}{3}$

so $|(f^3)'(p_1)| = 3^3 = 27 > 1$ so a periodic source.

2 (c) How many fixed points of f^2 are there in $[0, 1)$?

$$f^2(x) = 3(3x \pmod 1) \pmod 1 = 9x \pmod 1$$

fixed pts of f^2: $9x \pmod 1 = x$

so $9x = x + n$

$8x = n$ $\quad n = \{0, 1, 2, \cdots 7\}$

gives 8 solutions in $[0, 1) \Rightarrow$ 8 fixed pts.

2 (d) Prove that if an orbit $\{x_0, x_1, \ldots\}$ is eventually periodic, then x_0 is rational.

[Then the orbit $\{x_n, x_{n+1}, \cdots x_{n+k-1}\}$ is periodic for some n]

I cut the word 'eventually' in exam, so periodic x_0

$\Rightarrow f^k(x_0) = x_0$ $\quad$ ie $\quad 3^k x_0 \pmod 1 = x_0$

$\Rightarrow 3^k x_0 = x_0 + m$ $\quad$ for some $m \in \mathbb{Z}$.

$\Rightarrow x_0 = \dfrac{m}{3^k - 1} = \dfrac{\text{integer}}{\text{integer}} = $ rational.

cut in exam.

2 (e) Compute the Lyapunov *exponent* (not number) of such an [eventually] periodic orbit, and use this to estimate how many iterations will it take for an initial computer rounding error of 10^{-16} to reach size 1?

Lyapunov exponent of eventually periodic, asymptotically periodic, or merely periodic points is given by average over orbit points x_n of $\ln|f'(x_n)|$

Assuming $x = 1/3$ or $2/3$ is never hit, $f'(x) = 3$ always.

$\Rightarrow h = \ln 3$ is lyapunov exponent.

iterations, say k is this number. Then $3^k \cdot 10^{-16} \approx 1$

ln $\curvearrowleft$ $\quad k \ln 3 \approx 16 \ln 10$

$$k = \frac{16 \ln 10}{\ln 3} \approx 32$$

(e) BONUS: Derive a formula for the number of subintervals at level k.

[This is same as 2007 midterm 1]

You can spot sequence $\underset{(k=0)}{1}$ 2 3 5 8 .. Fibonacci.

Why?

At level k, i) # subintervals on left side = $\overbrace{\text{\# sub int. on Right at } k-1}^{\text{which by ii) is total at } k-2}$

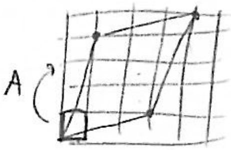
ii) # " " right = # total sub. int. on L∪R at k-1.

i.e. $F_k = F_{k-1} + F_{k-2}$, gives Fibonacci.

5. [6 points] Consider $T(\mathbf{x}) = A\mathbf{x} \pmod 1$, where $A = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$, acting on the torus $\mathbf{x} \in \mathbb{T}^2 = [0,1)^2$.

2 (a) Does the map T have an inverse? (explain using properties of the map)

No, since T is not one-to-one. Why not?



A maps unit square to something of area $|\det A| = |12-1| = 11$,

So there are many solutions to $T(\vec{x}) = \vec{c}$

3 (b) Find all fixed points of T in the torus.

$$3x + y = x \pmod 1 = x + m$$
$$x + 4y = y \pmod 1 = y + n$$

for some $n, m \in \mathbb{Z}$

ie $2x + y = m$

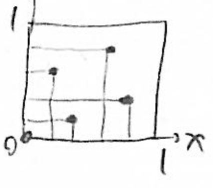
$x + 3y = n \rightarrow 2x + 6y = 2n$ $\}\xrightarrow{\text{subtract}}$ $5y = 2n - m \in \mathbb{Z}$

& $x = n - 3y = n - \frac{6}{5}n + \frac{3}{5}m$

$= -\frac{1}{5}n + \frac{3}{5}m$

n	m	y	x
0	0	0	0
0	1	$-\frac{1}{5} = \frac{4}{5}$ (mod 1)	$\frac{3}{5}$
0	2	$-\frac{2}{5} = \frac{3}{5}$	$\frac{6}{5} = \frac{1}{5}$
0	3	$-\frac{3}{5} = \frac{2}{5}$	$\frac{9}{5} = \frac{4}{5}$
0	4	$-\frac{4}{5} = \frac{1}{5}$	$\frac{12}{5} = \frac{2}{5}$

then it repeats...



How many expect?

$|\det(A - I)| = |6-1| = 5.$

1 (c) Answer (b) for the case of $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ ← diagonal so $x \rightarrow x \pmod 1$

$y \rightarrow 2y \pmod 1$

we know $y = 0$ is only fixed pt obeying this from 1d maps.

But all $x \in [0,1)$ is a fixed pt.

$\Rightarrow \{(x,y) : x \in [0,1), y = 0\}$.

this line.

[harder].

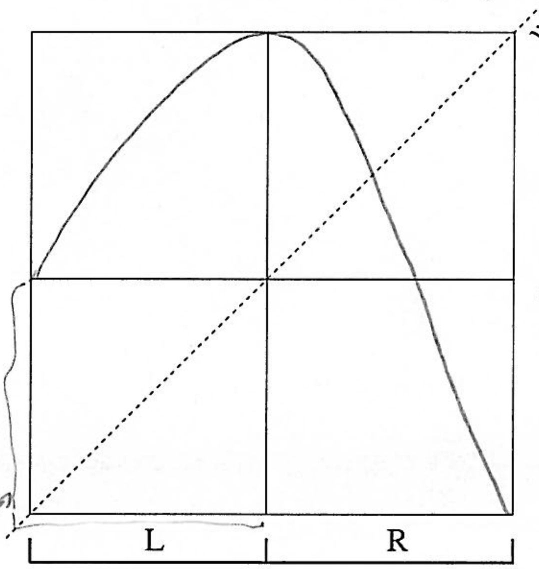
from transition graph, f must live in here  , and so $f(L) \supset R$

max of f must be top of R

4. [11 points]

2 (a) Draw a possible graph of a smooth continuous function f mapping $L \cup R$ to itself, with only one turning point, whose transition graph is that shown below. Use the axes and intervals shown to the right. (Be sure to check your f has the correct transition graph).

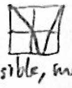
at $L \cap R$, I should have said!

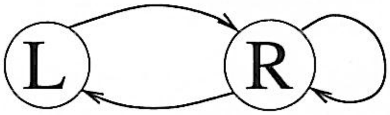


$f(R) = L \cup R$

$f(L) = R$

✓

 is also possible, messier!



no $L \to L$ allowed so f can't touch this square

2 (b) Prove that f has no fixed points in L.

Two ways: i) graph of f cannot enter the square in lower left.

or ii) a fixed pt in L would have itinerary $\overline{L}$, but L cannot follow L in transition graph.

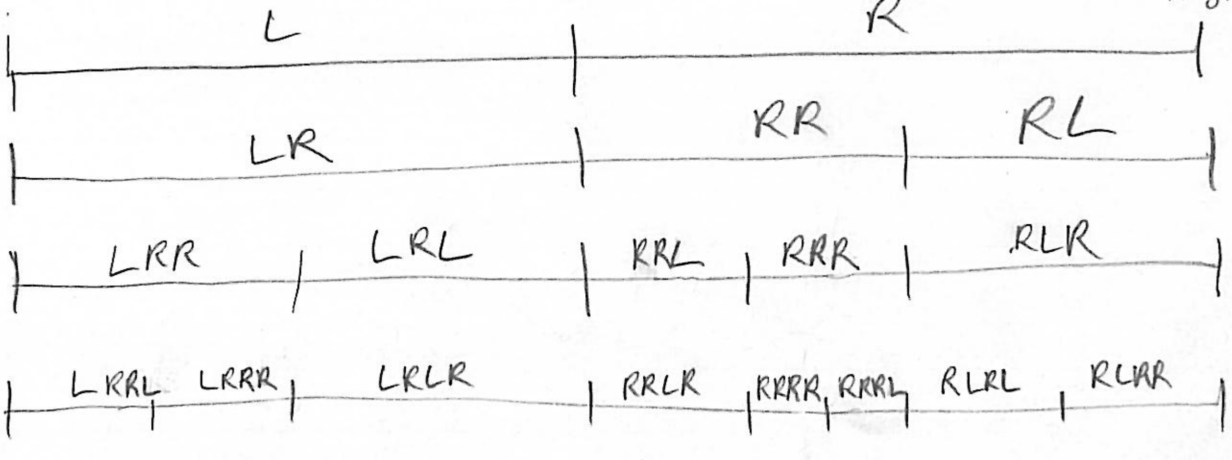
3 (c) Prove that there exist orbits which are not fixed, periodic, or eventually periodic.

By construction, write allowed orbits via transition graph, and they (as # letters → ∞) approach a point.

Eg LRLRRLRRRLRRRRL··· etc cannot be eventually periodic since otherwise its final letters would be a repeating string.

4 (d) Show the subdivision down to level 4 (that is, the correct ordering of all 4-symbol itinerary subintervals on $L \cup R$). Take plenty of horizontal space. How many subintervals are there? [Hint: you can answer the latter without the former]

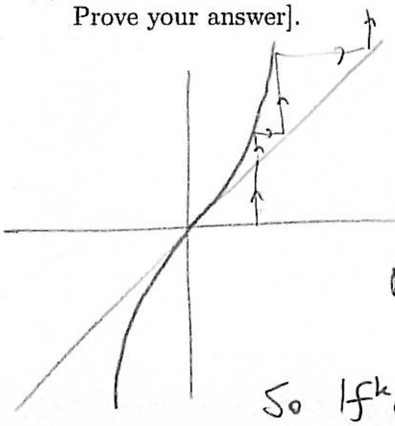
↳ 8 since that's how many length 4 strings are allowed in graph.



See worksheet: best way is by acting with f on whole groups of subintervals at once.

6. [8 points] Random short questions.

(a) The origin is a fixed point of $f(x) = \tan x$. Categorize it as a source, sink, or neither. [BONUS: Prove your answer].



since $\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$

$f'(0) = 1$ so theorem is not useful here.

However, cobweb plot shows it's a source.

Proof: $|\tan x| > |x| \quad \forall |x| < \pi/2$

(follows by geometry, e.g. 

are length vs heights).

So $|f^k(x)| > |f^{k-1}(x)| > \cdots > |x|$, so $\exists k$ st $|f^k(x)| > \delta$, no matter how small $|x|$ is.

(b) A map $f : \mathbb{R} \to \mathbb{R}$ has $f^6(x) = x$. What are the possible periods of x as a periodic fixed point, if any?

x may have periods 1, 2, 3, or 6.

(the divisors of 6).

(c) Give a precise mathematical definition of the *basin* of a fixed point p.

Basin of $p = \{ x : \lim_{k\to\infty} f^k(x) = p \}$

Note: no ε, no $N_\varepsilon(x)$, no maximal such set, etc...
It does require concept of limit.

(d) Explain in a sentence what a period-doubling bifurcation is (include a sketch of a bifurcation diagram with axes).

a transition of a period-k fixed point sink to a period-$2k$ sink, as a function of some parameter.