Barnett
11/16/07

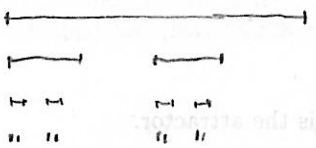
~ SOLUTIONS ~

Math 53: Chaos!: Midterm 2, FALL 2007

2 hours, 60 points total, 5 questions worth various points (proportional to blank space)

1. [16 points] Let $S \subset \mathbb{R}$ be the limit set produced in the following deterministic fashion: start with $[0,1]$ and repeatedly *remove* the 2^{nd} and 4^{th} quarter from each remaining interval.

[2] (a) Prove what the measure (total length) of the set S is.



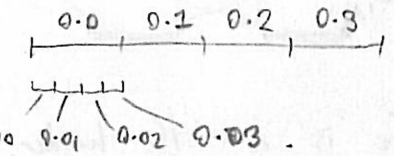
each iteration half the remaining length disappears

$\Rightarrow \text{meas}(S) = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$.

More formally: the iterations provide a sequence of covering intervals of vanishing total length.

[2] (b) Describe all points in S using base-4 ('quaternary').

numbers $0, \cdots 3$.



Any point with a 1 or 3 in the n^{th} digit is removed at the n^{th} iteration

$\Rightarrow S = \{x = 0.a_1a_2a_3\cdots : a_i = 0 \text{ or } 2, \text{ for all } i\}$

[2] (c) Find a rational in S [Hint: if stuck, part g) will help you].

but only if you got the right direction for the slope in $[\frac{1}{2}, \frac{3}{4}]$.

$0 = 0.\bar{0}$ is in S

$\frac{1}{2} = 0.2\bar{0}$ is in S

$\frac{2}{3} = 0.\bar{2}$ is in S ← $x = 0.\bar{2}$ so $4x = 2.\bar{2} = 2 + x$

Or any terminating or eventually-periodic sequence.

[2] (d) How many points are in S: finite, countably infinite, or uncountably infinite? Prove your statement.

$x = 0.200222\cdots$ is in S

We define a map from S to $[0,1]$ by converting all 2's in the quaternary expansion of $x \in S$ to 1's and interpreting as binary. Every real $y \in [0,1]$ (of which there are uncountably infinite) therefore has a distinct $x \in S$ [1] (in fact, some have two: eg $0.0\bar{2}$ and 0.2 both map to $y = 1/2$). QED.

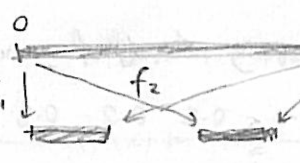
[3] (e) Find boxdim(S) (show your working).

The construction of S also supplies us with a recipe for covering with boxes of size b_n.

level		b_n	$N(b_n)$
$n=0$	0 ——— 1	1	1
1		$1/4$	2
2		$1/16$	4
3		$1/64$	8
$\vdots$		$\vdots$	$\vdots$
n		$\frac{1}{4^n}$	2^n

$$d = \lim_{n\to\infty} \frac{\ln N(b_n)}{\ln(1/b_n)} = \frac{n\ln 2}{n \ln 4} = \frac{\ln 2}{2\ln 2} = \frac{1}{2}$$

[3] (f) Describe a *probabilistic* game involving coin tosses and maps for which S is the attractor.



f_1 maps $[0,1]$ linearly to $[0, 1/4]$

so $f_1(x) = \frac{x}{4}$

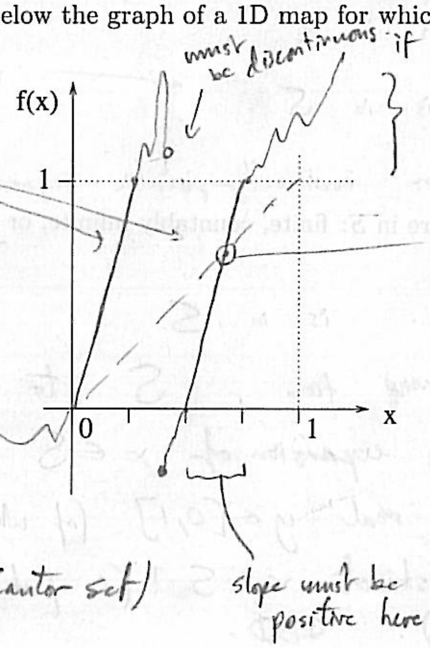
f_2 is a little harder, mapping $[0,1]$ to $[1/2, 3/4]$

so $f_2(x) = \frac{1}{2} + \frac{x}{4}$ will do (or $f_2(x) = 3/4 - \frac{x}{4}$)

Game: toss a coin & apply f_1 if heads, f_2 if tails.

[2] (g) Carefully sketch on the axes below the graph of a 1D map for which the set of points not in the basin of infinity is the set S.

f must be linear in $[0, 1/4]$ and $[1/2, 3/4]$



$f(x) \notin [0,1]$ for $1/4 < x < 1/2$

$f(x) > 0$ for $x > 3/4$

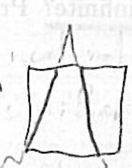
fixed point $x = 2/3$ is rational in S.

$f(x) < 0$ for $x < 0$

$f(x) \notin [0,1]$ for $1/4 < x < 1/2$ but otherwise no restriction.

slope must be positive here If the

Subtle point: the problem with



is that the order of the 4 quarters is flipped by the part of map in $[1/2, 3/4]$ and (unlike the middle-thirds Cantor set) S is not reflection-symmetric.

(h) BONUS: Give a modification of the original procedure which results in a 'fat fractal' (positive measure).

Simplest way is to remove smaller intervals whose total measure is < 1.

Eg: remove two intervals of length $\frac{1}{8}$ each, then four of length $\frac{1}{32}$, ... 2^n of length $\frac{1}{2^{2n+1}}$ each ...

Then no intervals remain but total measure is $1 - \frac{1}{4} - \frac{1}{8} - \cdots - \frac{1}{2^{n+1}} \cdots = \underline{\frac{1}{2}}$

2. [9 points]

[3] (a) Give a definition of box-counting dimension that only requires a discrete sequence of box sizes to be considered (be sure to include the condition on this sequence).

If $\{b_n\}$ is a sequence with $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$, $\lim_{n\to\infty} b_n = 0$.

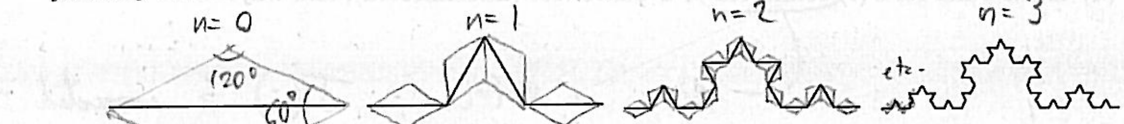
then $\text{boxdim}(S) = \lim_{n\to\infty} \frac{\ln N(b_n)}{\ln(1/b_n)}$ if limit exists.

(b) Could the super-exponentially decreasing sequence 2^{-n^2} be a valid sequence of box sizes?

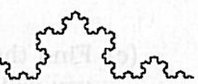
[1] $\lim_{n\to\infty} \frac{\ln 2^{-(n+1)^2}}{\ln 2^{-n^2}} = \lim_{n\to\infty} \frac{-(n+1)^2 \ln 2}{-n^2 \ln 2} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = 1$

So yes.

[5] (c) Find the box-counting dimension of the 'Koch curve' (a subset of $\mathbb{R}^2$) formed as shown by starting with a straight line segment then replacing the middle third of each straight line segment by the other two sides of the equilateral triangle. [Hint: describe your 'boxes'. To avoid colliding 'boxes' you may rotate them to cover without collisions]



Rhombus boxes rotated to cover line segments do not overlap.



$b_0 = 1 \qquad b_1 = 1/3 \qquad b_2 = 1/3^2 \quad \cdots \quad b_n = 1/3^n$

satisfies conditions on b_n since geometric (exponential) decay

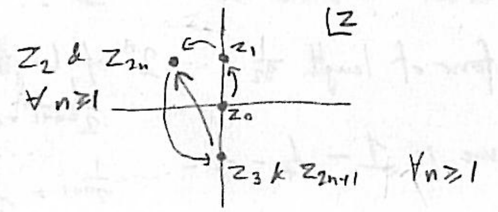
$N(b_0) = 1 \qquad N(b_1) = 4 \qquad N(b_2) = 4^2 \quad \cdots \quad N(b_n) = 4^n$

$\text{boxdim}(\text{Koch}) = \lim_{n\to\infty} \frac{\ln(4^n)}{\ln(3^n)} = \frac{\ln 4}{\ln 3} \approx 1.26$

clearly must be between 1 (line) & 2 (plane).

3. [15 points] Consider the complex map $z_{n+1} = z_n^2 + i$, where $i = \sqrt{-1}$.

[3] (a) Find, and describe as precisely as you can, the orbit of $z_0 = 0$.



$0 \longrightarrow i \longrightarrow i^2 + i \longrightarrow (-1+i)^2 + i \longrightarrow (-i)^2 + i$

$i^2 + i = -1 + i$

$(-1+i)^2 + i = -2i + i = -i$

$(-i)^2 + i = -1 + i$ Note has already appeared

Eventually periodic with period 2.

[3] (b) Give the mathematical definition of the Mandelbrot set M.

$M = \{ c : 0 \text{ is not in the basin of } \infty \text{ for map } z_{n+1} = z_n^2 + c \}$

[2] (c) Is i in M? (why?)

Yes, since $c = i$ results in a <u>bounded</u> orbit starting at $z_0 = 0$.

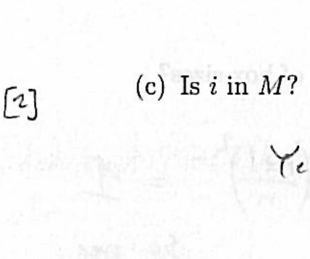
(its distance from 0 never exceeds $\sqrt{2}$)

[1] (d) Is the Julia set $J(i)$ connected or a Cantor set (disconnected), and why?

Since $i \in M$, $J(i)$ is connected.

(conversely $c \notin M \Rightarrow J(c)$ is a Cantor dust, disconnected).

[3] (e) Find the smallest closed disc you can which encloses $J(i)$ (the smaller the disc, the more points you will get!).



If disc radius a encloses J then all points outside disc, ie with $|z_0| > a$, have orbits growing without limit. So if you find an a such that $|z_n| > a \Rightarrow |z_{n+1}| > |z_n|$ then this radius a disc will do.

$c = i$ here

Triangle ineq: $|z_{n+1}| \geq |z_n|^2 - |c| = |z_n|^2 - 1$ so borderline case is $|z_{n+1}| = |z_n|$

Calling $x = |z_n|$ then $x = x^2 - 1$ ie $x^2 - x - 1 = 0$ ie $x = \frac{1+\sqrt{5}}{2} = \phi = 1.618\ldots$ golden ratio.

So if $|z_0| > \phi$, unbounded. <u>Radius ϕ encloses $J(i)$</u> (if you remember radius 2, got a point)

[3] (f) Is it possible that there could exist attracting periodic orbits not accounted for by what happened in part a)? Explain.

Fatou theorem: any basin of an attracting periodic orbit must include a critical point of the map $P_c(z) = z^2 + c$.

P_c only has the one critical point $z = 0$, and we already found in a) that this $z_0 = 0$ leads to the period-two orbit in a).

Therefore there cannot be another attracting periodic orbit.

can also do via complex 1d map stability.

(g) BONUS: Deduce the stability of the orbit in part a). What does this suggest about the measure of $J(i)$, and why?

In terms of real (x, y) with $z = x + iy$: $\vec{P}_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy + 1 \end{pmatrix}$ $\mathrm{Im}(c) = 1$

$c = i$

Jacobean

$$\vec{D}\vec{P}_i = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

So, period-2 $(\vec{p}_1 \vec{p}_2)$: stability given by eigenvalue of matrix product $\vec{D}\vec{P}_i(\vec{p}_1)\vec{D}\vec{P}_i(\vec{p}_2) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 4 & 4 \end{pmatrix}$

$\vec{p}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\quad \vec{p}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvalues

$$\begin{vmatrix} 4-\lambda & -4 \\ 4 & 4-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 32 = 0 \qquad \lambda = 4 \pm \sqrt{16 - 32} = 4 \pm 4i \quad \text{unstable.}$$

This suggests $\mathrm{meas}(J(i)) = 0$, which is true (it's a dendrite, connected but with fractal dim < 2)

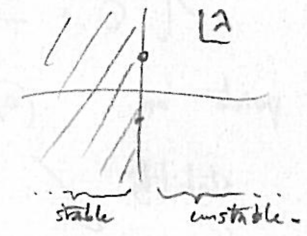
4. [7 points] Consider the nonlinear ODE system

$$x' = y - x(x^2 + y^2)$$
$$y' = -x - y(x^2 + y^2)$$

[4] (a) Analyse the stability of the fixed point $(0, 0)$ by linearization of the flow: explain exactly what can be concluded from the relevant theorem.

$$\vec{D}\vec{f} = \begin{pmatrix} -3x^2 - y^2 & 1 - 2xy \\ -1 - 2xy & -3y^2 - x^2 \end{pmatrix}$$

$$\vec{D}\vec{f}(\vec{0}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xrightarrow{\text{eigvals}} \lambda = \pm i$$



on borderline stability case. $(\mathrm{Re}\,\lambda = 0)$

$\Rightarrow$ cannot deduce anything by the nonlinear stability theorem.

[3] (b) By simplifying an expression for r', where $r^2 = x^2 + y^2$, state anything further you can prove about stability and asymptotic stability.

implicit derivative wrt. time

$$2rr' = 2xx' + 2yy'$$ using given ODEs

$$= 2x[y - xr^2] + 2y[-x - yr^2]$$

$$= \cancel{2xy} - \cancel{2xy} - 2(x^2+y^2)r^2$$

$$= -2r^4$$

So $r' = -r^3$ which is a 1d flow giving all positive r tending to $r = 0$ as $t \to \infty$

$\Rightarrow$ Both stable & asymptotically stable.

5. [13 points] Consider the nonlinear second-order ODE $\ddot{x} + 4x^3 - 2x = 0$. In this question be sure to think carefully about your *signs*.

$\frac{dP}{dx}$ so $P(x) = x^4 - x^2$

[1] (a) Write this as a first-order system.

use $x = x$
$y = x'$

so $\begin{cases} x' = y \\ y' = -4x^3 + 2x \end{cases}$

(b) Find all equilibrium points and either use linearization or the total energy function to prove as much about their stability as you can.

[3]

simultaneously $\begin{cases} 0 = y \\ 0 = -4x^3 + 2x \end{cases} \xrightarrow{\text{so}} 2x^2 = 1$ or $x = 0$

points are $(0,0)$ and $(\pm\frac{1}{\sqrt{2}}, 0)$

stability

stability

$\vec{Df}(\vec{0}) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ $\lambda = \pm\sqrt{2}$

saddle-point, unstable

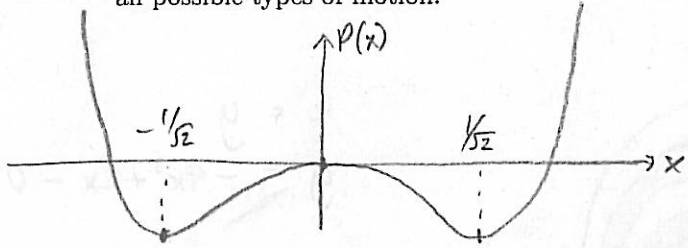
$\vec{Df}\begin{pmatrix} \pm 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -12x^2+2 & 0 \end{pmatrix}\Bigg|_{x=\frac{\pm1}{\sqrt{2}}} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$

$\lambda = \pm 2i$

BUT since nonlinear, cannot deduce stability. (Re $\lambda = 0$ for both).

Using total energy func $E(x,\dot{x}) = \frac{\dot{x}^2}{2} + P(x)$ you cannot leave a region bounded by contours of E, which proves these points are stable but not A.S.

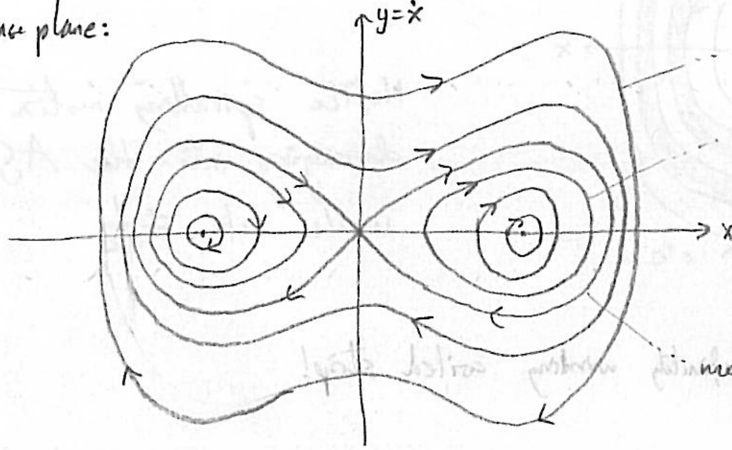
[3] (c) Graph the potential function and use this to sketch orbits in the phase plane $(x, \dot{x})$ which illustrate all possible types of motion.



$$\frac{dP}{dx} = 4x^3 - 2x \Rightarrow P(x) = x^4 - x^2$$

level curves $E(x,\dot{x}) = \frac{(\dot{x})^2}{2} + P(x)$ conserved

phase plane:



bounded periodic motion symmetric about $x=0$

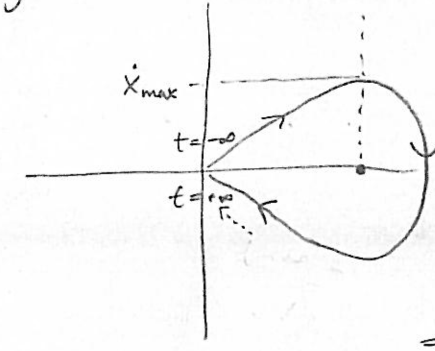
bounded periodic motion around $x>0$ only.

motion asymptotic to $(0,0)$ as $t \to \pm\infty$

[3] (d) What is the fastest speed ever reached anywhere in $t \in (-\infty, \infty)$ on an orbit that asymptotically approaches the middle (*i.e.* intermediate in x value) equilibrium point?

ie $x_0 = 0$

Eg take case of $x>0$ ($x<0$ mirror image):



The level curve of E reaching zero speed as $x \to 0$ has $E = \frac{0^2}{2} + P(0) = 0$

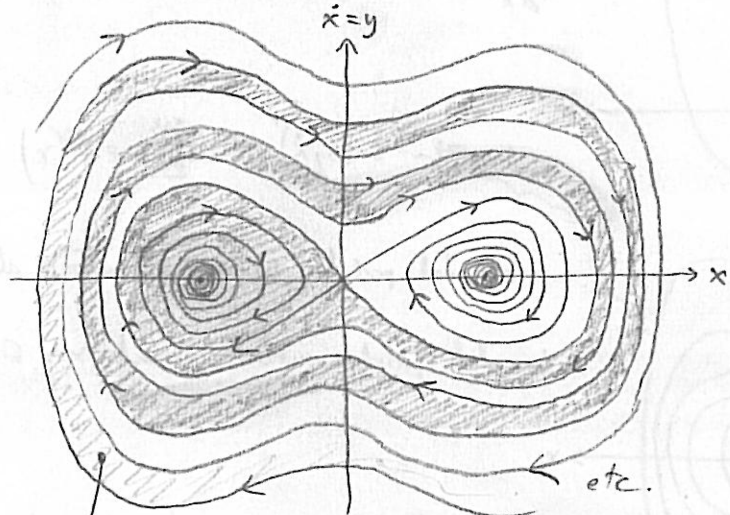
So at $x_0 = \frac{\pm 1}{\sqrt{2}}$, $P(x_0) = \frac{1}{2^2} - \frac{1}{2} = -\frac{1}{4}$

$$\Rightarrow 0 = \frac{(\dot{x}_{max})^2}{2} + P(x_0)$$

$$x_{max} = \sqrt{2\left(\tfrac{1}{4}\right)} = \frac{1}{\sqrt{2}}$$

damping with $c = 0.1 > 0$

[3] (e) On new axes draw phase-plane orbits of the *modified* ODE $\ddot{x} + 0.1\dot{x} + 4x^3 - 2x = 0$. Carefully shade in and indicate the shape and structure of the *basin* of the left-most (*i.e.* with smallest x value) equilibrium point.



$$\dot{x} = y$$
$$\dot{y} = -4x^3 + 2x - 0.1y$$

Notice spiralling motion decaying into the AS wells at $\pm\frac{1}{\sqrt{2}}$

basin of $x_0 = -1/\sqrt{2}$: an infinitely winding coiled strip!