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A Single Agent Might Make a Difference, … but so does a single variation in our model

1. Introduction

The well-known economic "Cobweb Model"[1] explains cyclical market fluctuations as a result of the time lag in the producers' response to a change in prices. Producers have to plan their future production (x_{t+1}) ahead of time, basing it only on current production (x_t) and prices (p_t). Thus, for some supply function *f*,

$$x_{t+1} = f(x_t, p_t)$$

Prices, on the other hand, are determined by current output. Hence for some demand function *g*,

$$p_t = g(x_t) \text{ and thus } x_{t+1} = f\big(x_t, g(x_t)\big),$$

defining a discrete mapping.

In their essay, "Stability, chaos and multiple attractors: a single agent makes a difference," Onozaki et al. specify two different simple supply functions and one demand function to prove mathematically that adding a single agent to a large market consisting only of producers of the other supply function can change the qualitative behavior of the market. To investigate into Onozaki et al.'s theory, I wrote a Matlab program that evaluates their model and various alterations numerically. We will first use this program to illustrate Onozaki et al.'s argument that generalizations about the supply function easily misrepresent market behavior. We then use the program to show that the simplifications of the model reduce the model's accuracy at least as much.

2. Onozaki et al.'s Model

For a given price *p*, producers choose the output that maximizes their profits Π, thus

$$\max_x \Pi(x) = px - C(x)$$
$$C'(x) = p$$
$$x = C'^{-1}(p)$$

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[1] Discussed as early as in 1938 in Mordecai Ezekiel's *The Cobweb Theorem*

2.1 Naïve Optimizers

For simplicity, Onozaki et al. assume the same quadratic cost function C for all producers. They then define the "naïve optimizer" producer to determine his future output x_{t+1}, basing it solely on current prices p_t, and thus:

$$C(x) = a\frac{x^2}{2}$$

$$\text{so} \quad ax_{t+1} = p_t$$

$$x_{t+1} = \frac{p_t}{a}$$

They combine this supply function with an isoelastic[2] demand function $p = g(X)$; that is, for any total X produced a marginal change in X will lead to the same marginal change in p:

$$g(X) = \frac{b}{X^\sigma}, \text{ for } \sigma \geq 0.$$

And thus a market consisting entirely of n naïve optimizers has the following discrete mapping:

$$X_{t+1} = \sum_{i=1}^{n} x_{i_{t+1}} = n\left(\frac{p_t}{a}\right) = \frac{n}{a}\left(\frac{b}{X_t^\sigma}\right)$$

$$\text{let } \frac{nb}{a} = k, \text{ and so } X_{t+1} = \frac{k}{X_t^\sigma}$$

Consequently, if $\sigma < 1$, X will converge to the stable equilibrium $X = \sqrt{k}$, if $\sigma = 1$, for all X we will have stable period-2 orbits and for $\sigma > 1$ the output cycles will explode:[3]

Fig1: σ = 0.8

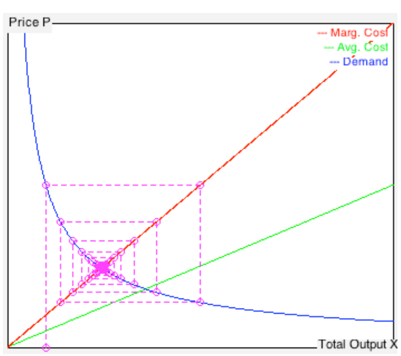


Fig2: σ = 1

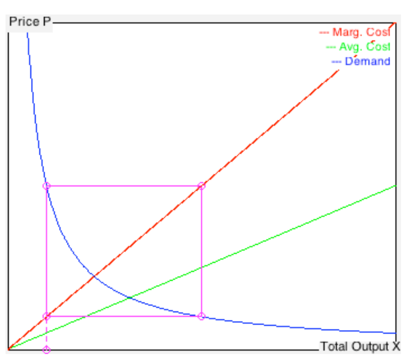
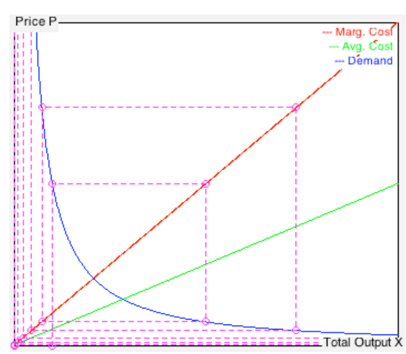


Fig3: σ = 1.2



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[2] Constant elasticity, as elasticity $= -\left(\frac{dg}{dx}\times\frac{x}{g}\right)^{-1} = \sigma^{-1}\left(\frac{b}{x^{\sigma+1}}\times\frac{xx^\sigma}{b}\right) = \sigma^{-1}$

[3] Which you can test using the model graph of the program, changing inv. el. above the graph and selecting the cobweb plot of agent 1 or simply proof, as demand is isoelastic and supply is linear.

2.2 Cautious Adapters

For the second type of producers, Onozaki et al. introduce the "cautious adapter" that chooses to adopt this optimum based on current prices only partially, such that, for $c \in [0,1)$,

$$x_{t+1} = x_t + c(\tilde{x} - x_t), \text{ where } \tilde{x} = \frac{p_t}{a} \text{ (the quantity the n.o. choose).}$$

So for a market consisting only of n cautious optimizers with the same c, we have

$$X_{t+1} = n(x_t + c(\tilde{x} - x_t)) = n\left(x_t + c\left(\frac{b}{aX_t^\sigma} - x_t\right)\right) = X_t + c\left(\frac{nb}{aX_t^\sigma} - X_t\right)$$

$$= X_t + c\left(\frac{k}{X_t^\sigma} - X_t\right)$$

as $nx_t = X_t$. Onozaki et al. showed in an earlier paper[4] that this mapping converges to $X = \sqrt{k}$ if $\sigma < (2-c)/c$, at $\sigma = (2-c)/c$, the fixed point undergoes period doubling, for greater σ the map has one attracting periodic or chaotic orbit. Choosing $c = 0.5$, we use the program to generate the bifurcation diagram of X versus σ:

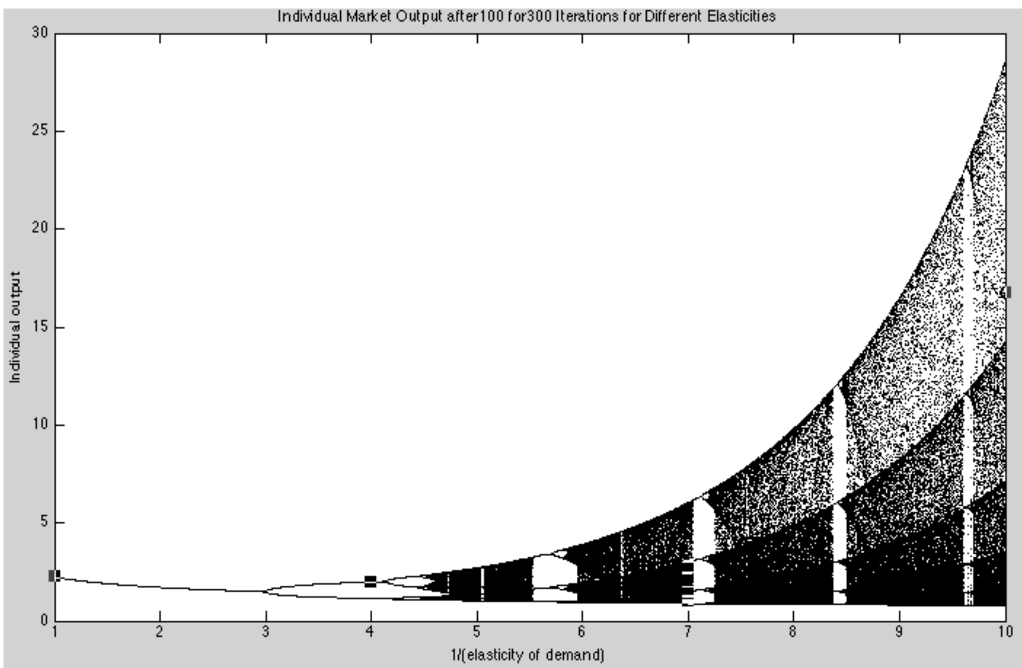

Fig4 (The first period doubling happens at σ = 3 = (2 - .5)/.5, as Onozaki et al. predict)

We can use the program to investigate further, noticing that for smaller c, σ needs to be higher for period doubling and higher periodic orbits to occur, which follows our intuition, as the 'more-cautious' adapters need a greater variation in price to change their output significantly.

[4] Onozaki et al. (2000)

2.3 Different Adjustment Strategies in one Market

Suppose we have a market with n naïve optimizers that each produce u and m cautious adapters with the same $c \in [0,1)$ that each choose to produce v, then from before, we have

$$\begin{bmatrix} u_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} \dfrac{b}{aX_t^\sigma} \\ v_t + c\left(\dfrac{b}{aX_t^\sigma} - v_t\right) \end{bmatrix} \therefore \begin{bmatrix} nu_{t+1} \\ mv_{t+1} \end{bmatrix} = \begin{bmatrix} n\dfrac{b}{aX_t^\sigma} \\ m\left(v_t + c\left(\dfrac{b}{aX_t^\sigma} - v_t\right)\right) \end{bmatrix}$$

where $X_t = nu_t + mv_t$. We can scale this equation such that $n+m = 1$ with m $\in (0,1)$, moreover, we let $k = b/c$ and so we have

$$\begin{bmatrix} (1-m)u_{t+1} \\ mv_{t+1} \end{bmatrix} = \begin{bmatrix} (1-m)\dfrac{k}{((1-m)u_t + mv_t)^\sigma} \\ m\left(v_t + c\left(\dfrac{k}{((1-m)u_t + mv_t)^\sigma} - v_t\right)\right) \end{bmatrix}$$

Letting $x = (1-m)u$ and $y = mv$, we can define the map F to iterate the above equation such that

$$F(x,y) = \left(\frac{(1-m)k}{(x+y)^\sigma}, y + c\left(\frac{mk}{(x+y)^\sigma} - y\right)\right)$$

Suppose m = 0.01, and c = 0.5 (one cautious adapter in a market of naïve optimizers). We use the program to find the bifurcation diagram for σ from 0 to 2.5:

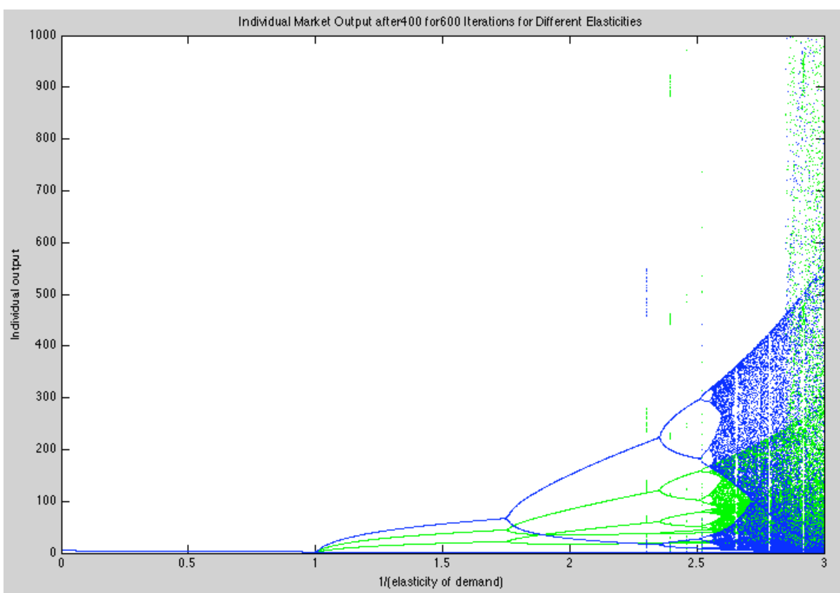

Fig5 (blue: naïve optimizers, green: cautious optimizers – both scaled by their inverse frequency)

Even though as σ increases, the maximum output still grows very fast (after σ = 2.8, output reaches 70,000), the price no longer explodes, but is bound. The cautiousness of the 0.01 producers prevents output from approaching zero and thus puts a limit on the maximum the price can achieve.

In fact, in their preposition 2, Onozaki et al. prove the above intuition, proving that for our map F, "for $m \in R^2_{++}$, every trajectory starting from R^2_{++} is trapped into a compact region in R^2_{++}." Instead of exploding, trajectories get attracted to periodic or chaotic orbits.

Suppose in the market of cautious adapters we discussed before, one naïve optimizer enters, such that m = 0.99:

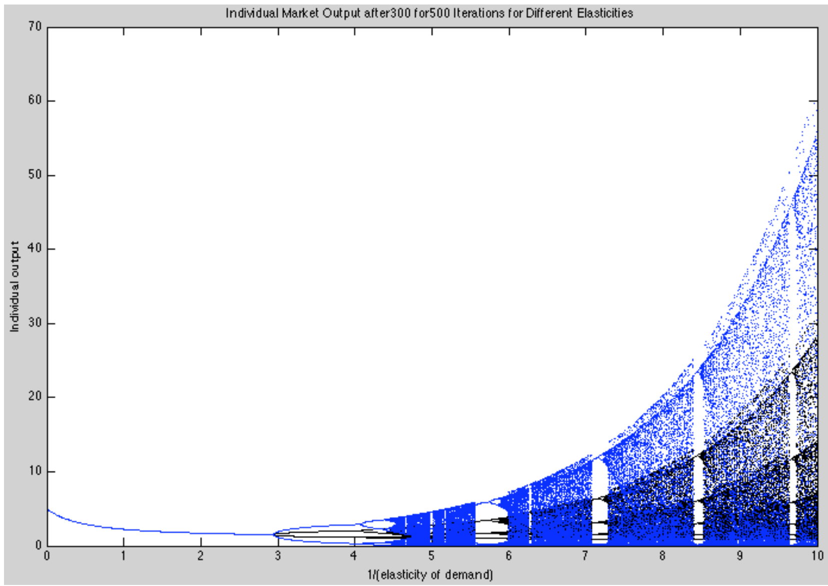

Fig6 (black: cautious adapters, blue: naïve optimizers – output scaled by inverse frequency)

In comparison to Figure 3, the clean boundary of the cascades gets broken; even more interesting Onozaki et al. prove that whereas for m = 1, ∃ exactly one attractor, for m < 1 but sufficiently close to 1, for specific σ, different periodic attractors coexist.[5]

3. The Matlab Program

Rather than reviewing Onozaki et al.'s theory further, we will investigate numerically using a Matlab program that allows us also to modify the model instead of just changing the heterogeneity of producer strategies.

To run the program you will need to load both the m-file econ.m and the fig-file econ.fig into your current directory of Matlab. Typing econ into your command window will launch the GUI as well as the bifurcation diagram for price vs. elasticity (launching may take a few seconds). The GUI shows you a price versus output graph of your currently selected model, showing the marginal cost and the average cost for producers, as well as the demand function for a certain Inv. El. that you can specify above. You can change these to the right of the model graph. The graph will be immediately updated if you choose a different function from the pop-up menu or change on of its variables below.

[5] see Onozaki et al. (2003) Proposition 3

Under the graph you can modify the producer strategies of four different agents, the first three agents can be modified in the same way (influence their *c* value); however the red rational agent has a different adaptation strategy that you cannot modify (more about this later). Selecting one of the radio buttons below the agents' variables will add a cobweb plot to the model graph, modeling the behavior of the agent if he is alone in the market facing the chosen inverse elasticity.

In the bottom right box you can choose the number of iterations and the values of the elasticity that you want to evaluate. By checking or unchecking Price, Indv. Output, and/or Profit, you can choose which graphs to open or close. Graphs will not update immediately, as iterating all your chosen values can be timely. Instead they will always display the results from the last iteration. To update your iteration press the Update button. It will display when the program is busy calculating – any events you execute during these calculations will be delayed and executed as soon as the iteration is done.

If you would like to know more on the functions and actions behind the GUI, please have a look at the commented code of econ.m. As econ.m uses vectorization on a three-dimensional array to compute its calculation intensive routines, Matlab can compute a multitude of iterations for different σ values very rapidly; still for the more complicated models (especially for models in which producers remember prices for numerous periods) iterations may take longer, in which case it is advisable to choose a greater step size for σ (= Inv. Elasticity on the GUI) and fewer iterations, at least if you are running a routine for the first time, not knowing its runtime.

4. Investigating beyond Onozaki et al.

For this section, we will look at numerous numerical results. Thus, instead of providing graphs, I will give instructions on how to produce these graphs with econ.m. Start econ.m. After the GUI has appeared change the supply function to S3. This function alters Onozaki et al.'s supply function by shifting its cost function by a minimum fixed cost (minC) and a minimum output (minX). Consequently, the marginal cost curve shifts to the right (see model graph). You can click on the radio button bellow Agent one to see that now also at an inv. elasticity of one, agent one's trajectory is attracted by the fixed point. Update the graph. As you see no more chaos, you may want to change the range of inv. elasticity to 0 to 100 – make sure to change the step size to 0.05. To understand why no more chaos occurs change the inv El. of the model graph to 100. Consequently, the almost vertical demand is at $X = 1$. Discarding the first 3 iterations of the cobweb plot, you can see that even agent 1 gets drawn to the stable equilibrium at $X = 1.5$ and p = 0.

As we did not get chaos because minX > 1 and thus demand for high σ, we lower minX (, and minC) to 0.5. We change the inv. El. of our model graph to 7 and choose the cobweb plot for agent 2, who settles in a period-4 orbit (choose 200 iterations an discard 150 for the cobweb model). If we look at agent 1, however, we seem to have a period two orbit, that oscillates between very high price and very high output. We choose inv. elasticity from 0 to 7 with step size 0.005 and update our graph to get a bifurcation diagram, that seems to indicate chaos again.

One of the reasons for chaos seems to be that the price of our naïve optimizer can get extremely high for high σ, causing great shifts in X_{t+1}, the infinite prices that are made possible by

Onozaki et al.'s isoelastic demand function are unrealistic though. To adjust for this, we switch to demand function D4 which shifts Onozaki's demand function in by xOff and pOff, creating finite limits for X and p. The large period two cycle of agent 1 is once again reduced to fit on the model graph. However, updating our bifurcation graph for inverse elasticity from 0 to 10 shows that chaotic behavior / large periodic orbits were only cancelled for smaller σ, for σ > 8.5 there still seems to be chaos.

Rather than playing around further with different demand and supply functions,[6] we introduce the concept of memory, i.e.

$$x_{t+1} = f\left(x_t, p_t, p_{t-1}, \ldots, p_{t-n}\right)$$

In the "Different Agents" we change the pop-up menu to remember n periods (n=3), such that agents now take the average price of the last three periods to determine their future optimum. As this more receptive to old periods, this might nullify our cautious adapter agent 2 – so we change the frequencies of Agent 1 to 1 and of Agent 2 to 0. We change the total iterations to 500 and discard to 300, we evaluate inv. elasticity from 0 to 50 with step size 0.05. Then, for σ > 4.7, we have a period–2 attractor, just as the cobweb model predicts. Suppose, however, we justify the cautious adapter by acknowledging for many firms adjusting is a gradual process. Changing the frequencies to 0.5 for both again, gives us seemingly chaotic orbits for σ > 15 within the bounds of 0 and 2.25.

Now choose S5. For this function you can modify its power. More importantly, however, this function first has increasing than decreasing returns to scale, that is

$$x = \begin{cases} -\left|\left(\min X - x\right)\right|^{power} & if : x < \min X \\ +\left(x - \min X\right)^{power} & if : x \geq \min X \end{cases}$$

The marginal cost function thus becomes more interesting and more related more common economic models.[7] Because marginal cost first decreases, producers now have a second restraint for production, that is, MC' = C'' > 0, else they produce 0. Moreover, change memory to infinite, step size to 0.02, and Discard to 450. Updating gives us a single line that slopes down and hits the x-axis at about 10. If we change Discard to 650 and total iterations to 700, however, this x intercept moves further to the right. So at least the infinite memory seems to eventually move us to a stable equilibrium.[8]

The GUI lets you manipulate a lot of other things too, if you are interested. A few extra features such as the rational agent or the over adjuster agent 3 (btw. you can change his cautiousness as you can for agent 1 and 2) produce quite a number of intriguing results especially if we take the average profit graph into account.

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[6] You are however encouraged to do so by yourself. To get more documentation to the actual functions please have a look at the commented code.

[7] At first the producer gets increasing returns to scale, i.e. as he produces more he can make production more specific and thus efficient. At some point however, more specialization is not possible and increases in scale rather present managerial challenges; moreover resources might become scarce and thus marginal cost increases again.

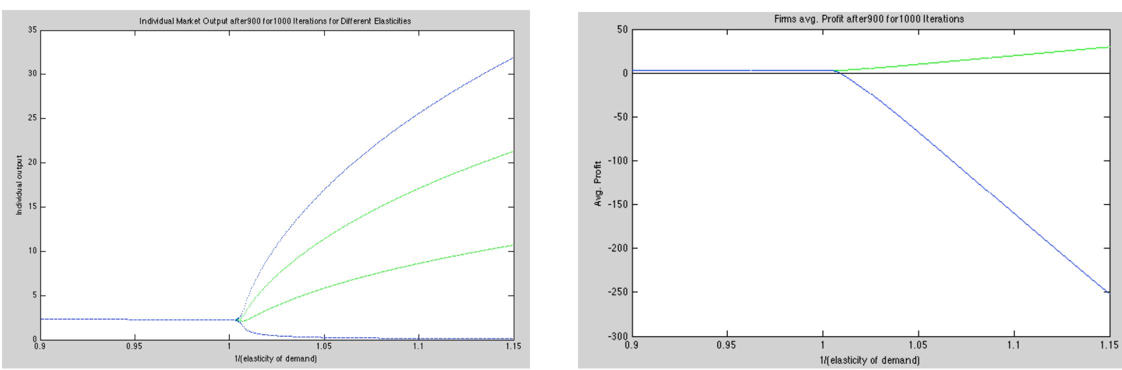
[8] There is a way of avoiding that however, too – the discontinuous supply function can work out to no equilibrium at all!

5. No Explosion, nor any naive optimizers in inelastic markets

We used the variations of the program on Onozaki et al.'s much more simplified model to investigate a more comprehensive problem. While the program in that regard allowed for more complex as well as more stable outcomes than the simplified model, we did not use the model to study Onozaki et al.'s main topic the heterogeneity in producers. However, one of the program's features very directly illustrates a major problem Onozaki et al. did not recognize in their theory.

Onozaki et al. did not consider the profits or losses that the different agents make, which we can illustrate them easily using the average profit graph. The key idea is that a company that runs a loss on average will eventually use up their reserves and thus go bankrupt or if it realizes its problem early enough change its strategy. Consequently, explosions are not possible, as the increasing price that firms receive during the cyclic period that they produce less and less in, will eventually be not enough to cover the quadratic cost of producing more and more.[9]

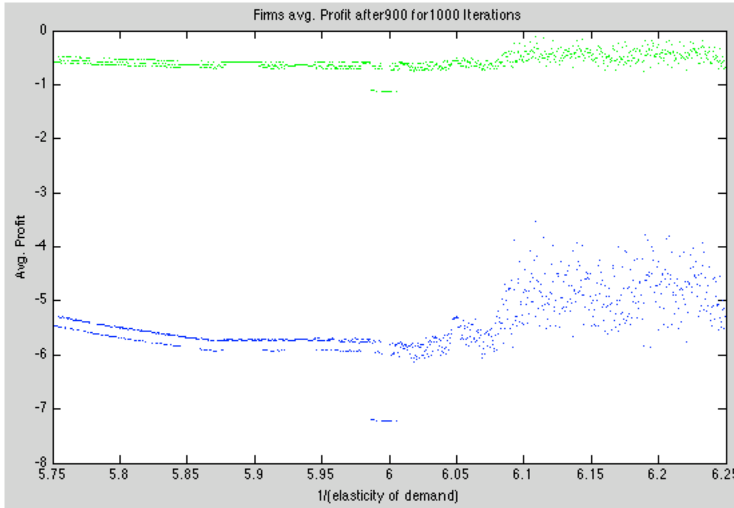
Similarly, a market that consists mainly of naïve optimizers will also not be "stabilized" by a single cautious optimizer as Onozaki et al. phrase it. As a single cautious optimizer is most unlikely to dampen the fluctuations in price enough, such that the naïve optimizers do not make a negative profit on average. The graphs below show the enlarged part of the bifurcation diagram the average profit for our previous example with m = 0.01, and c = 0.5:



(blue: naïve optimizer, green: cautious optimizer, all data called by individual inverse of frequency)

And finally, naïve optimizers cannot (or at least almost never) enter fluctuating markets of high σ, as the high σ will make them fluctuate much more than the cautious adapters and thus lose much more in periods where output is too high and win much less, when output is low. Yet, for the naïve optimizer to create coexisting periods of periodic attractors σ must be high. For illustration, for the only example Onozaki et al. give to show two coexisting periodic attractos (c = 0.5, σ = 6, m = 0.961) we show the profit graph below.

[9] As X increases linear with the high price (as we set linear marginal cost equal to price) but the cost function is quadratic and thus increases faster.



(blue: Profit of naïve optimizer, necessary for coexisting orbit)

On the other hand the program shows that chaos can exist also in profitable markets. Whether the model leads to chaos or a stable solution can be quite dependent on single variables of our models (eg. minX) which hints at possible employment of appropriate models for government policy or choosing the right production strategy.

We have only employed very little of the modeling options, the Matlab program offers us. Moreover, the code of the program is easily modifiable to allow for even more comprehensive models. Thus if you're interested use it to explore more chaotic economics.

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