Math 53: Chaos!: Midterm 2, FALL 2007

2 hours, 60 points total, 5 questions worth various points (proportional to blank space)

1. [16 points] Let $S \subset \mathbb{R}$ be the limit set produced in the following deterministic fashion: start with $[0, 1]$ and repeatedly *remove* the 2^{nd} and 4^{th} quarter from each remaining interval.

   (a) Prove what the measure (total length) of the set S is.

   (b) Describe all points in S using base-4 ('quaternary').

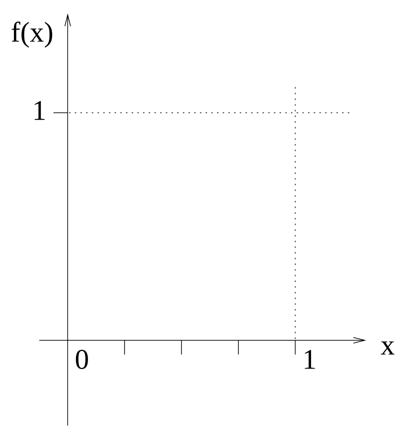
   (c) Find a rational in S [Hint: if stuck, part g) will help you].

   (d) How many points are in S: finite, countably infinite, or uncountably infinite? Prove your statement.

(e) Find boxdim(S) (show your working).

(f) Describe a *probabilistic* game involving coin tosses and maps for which S is the attractor.

(g) Carefully sketch on the axes below the graph of a 1D map for which the set of points not in the basin of infinity is the set S.



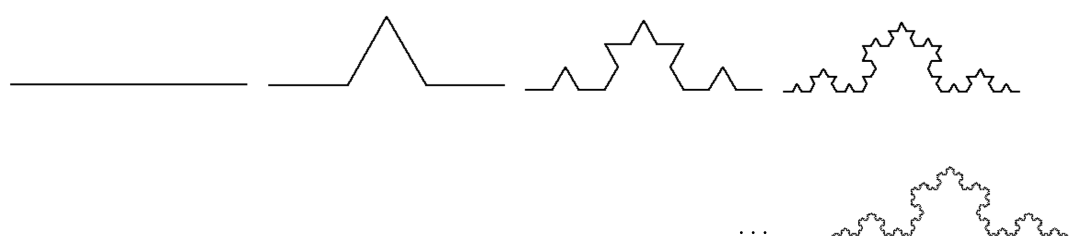
(h) BONUS: Give a modification of the original procedure which results in a 'fat fractal' (positive measure).

2. [9 points]

(a) Give a definition of box-counting dimension that only requires a discrete sequence of box sizes to be considered (be sure to include the condition on this sequence).

(b) Could the super-exponentially decreasing sequence 2^{-n^2} be a valid sequence of box sizes?

(c) Find the box-counting dimension of the 'Koch curve' (a subset of $\mathbb{R}^2$) formed as shown by starting with a straight line segment then replacing the middle third of each straight line segment by the other two sides of the equilateral triangle. [Hint: describe your 'boxes'. To avoid colliding 'boxes' you may rotate them to cover without collisions]



3. [15 points] Consider the complex map $z_{n+1} = z_n^2 + i$, where $i = \sqrt{-1}$.

(a) Find, and describe as precisely as you can, the orbit of $z_0 = 0$.

(b) Give the mathematical definition of the Mandelbrot set M.

(c) Is i in M? (why?)

(d) Is the Julia set $J(i)$ connected or a Cantor set (disconnected), and why?

(e) Find the smallest closed disc you can which encloses $J(i)$ (the smaller the disc, the more points you will get!).

(f) Is it possible that there could exist attracting periodic orbits not accounted for by what happened in part a)? Explain.

(g) BONUS: Deduce the stability of the orbit in part a). What does this suggest about the measure of $J(i)$, and why?

4. [7 points] Consider the nonlinear ODE system

$$x' = y - x(x^2 + y^2)$$
$$y' = -x - y(x^2 + y^2)$$

(a) Analyse the stability of the fixed point $(0, 0)$ by linearization of the flow: explain exactly what can be concluded from the relevant theorem.

(b) By simplifying an expression for r', where $r^2 = x^2 + y^2$, state anything further you can prove about stability and asymptotic stability.

5. [13 points] Consider the nonlinear second-order ODE $\ddot{x} + 4x^3 - 2x = 0$. In this question be sure to think carefully about your *signs*.

   (a) Write this as a first-order system.

   (b) Find all equilibrium points and either use linearization or the total energy function to prove as much about their stability as you can.

(c) Graph the potential function and use this to sketch orbits in the phase plane $(x, \dot{x})$ which illustrate all possible types of motion.

(d) What is the fastest speed ever reached anywhere in $t \in (-\infty, \infty)$ on an orbit that asymptotically approaches the middle (*i.e.* intermediate in x value) equilibrium point?

(e) On new axes draw phase-plane orbits of the *modified* ODE $\ddot{x} + 0.1\dot{x} + 4x^3 - 2x = 0$. Carefully shade in and indicate the shape and structure of the *basin* of the left-most (*i.e.* with smallest x value) equilibrium point.