MATH 53 WORKSHEET : Rate of volume change in flow

Barnett 11/28/07

$\boxed{n=2 \text{ case}}$

Jacobean $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of time-t map

satisfies the ODE $\dot{J} = (\vec{D}\vec{f})\, J$ ← Jacobean of flow.

Ie $\begin{bmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{bmatrix} = \overbrace{\begin{bmatrix} w & x \\ y & z \end{bmatrix}}^{\vec{D}\vec{f}} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Find $\frac{d}{dt}(\overbrace{\det J}^{\text{volume element under flow.}})$ in terms of a, b, c, d, w, x, y, z.

write $\det J$ & use product rule,

Simplify (half the terms will cancel):

Factorize into $\frac{d}{dt}(\det J) = (\quad ? \quad)\cdot \det J$

something depending on $\vec{D}\vec{f}$

What is the "something" in terms of partials of $\vec{f}$?

its multivariable calculus name is ?

Write the (general n case) : $\frac{d}{dt}(\det J) =$ ← $\boxed{\text{Liouville's Theorem}}$

Evaluate this for Hamiltonian flow $\dot{\vec{z}} = \underbrace{\left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right] \vec{\nabla}_z H(\vec{z})}_{\text{this is } \vec{f}(\vec{z}).}$

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~ SOLUTIONS ~

$\boxed{n=2 \text{ case}}$ Jacobean $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of time-t map

satisfies the ODE $\dot{J} = (\vec{D}\vec{f})\, J$ ← Jacobean of flow.

Ie $\begin{bmatrix} \dot a & \dot b \\ \dot c & \dot d \end{bmatrix} = \overbrace{\begin{bmatrix} w & x \\ y & z \end{bmatrix}}^{\vec{D}\vec{f}} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Find $\frac{d}{dt}(\overbrace{\det J}^{\text{volume element under flow}})$ in terms of a, b, c, d, w, x, y, z

write $\det J$ & use product rule,

$\frac{d}{dt}(ad - bc) = \dot a d + a\dot d - \dot b c - b\dot c$ substitute $\dot a = wa + xc$ etc.

$= wad + \cancel{xcd} + \cancel{ayb} + azd - wbc - \cancel{xdc} - \cancel{bya} - bzc$

Simplify (half the terms will cancel):

$= ad(w+z) - bc(w+z)$

$= (w+z)\underbrace{(ad-bc)}_{\det J}$

Factorize into $\frac{d}{dt}(\det J) = (\overbrace{w+z}^{\mathrm{Tr}(\vec{D}\vec{f})}) \cdot \det J$ ~ something depending on $\vec{D}F$

trace of matrix.

What is the "something" in terms of partials of $\vec{f}$? $\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} = \operatorname{div}\vec{f}$

$\vec{\nabla}\cdot\vec{f} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial z_i}$ its multivariable calculus name is ? $\vec{\nabla}\cdot\vec{f}$ divergence

Write the (general n case) : $\frac{d}{dt}(\det J) = (\vec{\nabla}\cdot\vec{f}) \det J$ ← $\boxed{\text{Liouville's Theorem}}$

Evaluate this for Hamiltonian flow $\dot{\vec{z}} = \underbrace{\left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right] \vec{\nabla}_z H(\vec{z})}$ this is $\vec{f}(\vec{z}) = \begin{bmatrix} \partial H/\partial p_1 \\ \vdots \\ \partial H/\partial p_n \\ -\partial H/\partial q_1 \\ \vdots \\ -\partial H/\partial q_n \end{bmatrix}$ (2n components)

$\vec{\nabla}\cdot\vec{f} = \underbrace{\frac{\partial}{\partial q_1}\frac{\partial H}{\partial p_1} + \cdots + \frac{\partial}{\partial q_n}\frac{\partial H}{\partial p_n} - \frac{\partial}{\partial p_1}\frac{\partial H}{\partial q_1}}_{\text{cancel}} - \cdots - \frac{\partial}{\partial p_n}\frac{\partial H}{\partial q_n} = 0$ etc

so $\underline{\det J = \text{const.}}$ QED.