# Math 50: Midterm 2

65 minutes, 70 points. No algebra-capable calculators. Try to use your calculator minimally—you barely need it. Show working/reaonsing, since only that way could you get partial credit.

1. [10 points] IQ (the supposed 'intelligence quotient') is an integer scale designed to be normally-distributed in the population, with $\mu = 100$ and $\sigma = 15$.

   (a) What fraction of the population is then required to be a 'moron' (a technical term, defined by $50 < \text{IQ} < 75$) ?

   (b) What is the chances that the average IQ of a random sample of size 25 of the population has an IQ of at least 106? [Hint: IQ is an integer quantity; but you will not lose much for ignoring this]

2. [12 points] 1000 people randomly sampled from the US population are given a survey asking if they are in favor of gay marriage.

   (a) Suppose 750 of the 1000 are in favor. Construct a 95% confidence interval on $p$, the fraction of the US population that are in favor.

   (b) Suppose $p$ is unknown and you want to design a survey to estimate $p$ with a *margin of error* of 3%. What is the minimum number of people you need to survey?

3. [23 points] Data are drawn from the model pdf

$$f_Y(y;\theta) = 2y/\theta^2 \quad \text{for} \quad 0 < y < \theta, \quad \text{zero otherwise.}$$

Given samples $\{y_1, \ldots, y_n\}$, we wish to estimate the parameter $\theta$.

(a) Find the Method of Moments estimator $\hat{\theta}$.

(b) Is this estimator unbiased? (Prove your answer)

(c) What is the efficiency of this estimator, $\text{Var}(\hat{\theta})$ ?

(d) As with the uniform pdf, the Maximum Likelihood estimator is $\hat{\theta}_{ML} = Y_{max}$. What is the bias of this estimator? If needed, suggest a fix which makes it unbiased.

(e) Prove whether the estimator $\hat{\theta}_{ML}$ is consistent or not.

(f) Give an example of an estimator which is *not* consistent (either for the above pdf, or any pdf of your choosing).

4. [11 points] A coin of unknown bias $0 \leq p \leq 1$ is flipped 3 times and gives the data: heads, tails, heads.

(a) Assuming an uninformative prior, compute the (correctly-normalized) posterior pdf on $p$ given this data.

(b) Given this posterior, compute $P(p \leq 1/2)$, that is, the Bayesian answer to the question, "what is the chance that the coin is biased in the tails direction?"

5. [14 points] Some distributions, such as those of salaries or earthquake strengths, can be modeled by a power-law pdf with parameter $\theta > 0$, thus

$$f_Y(y;\theta) = \theta y^{-1-\theta}, \quad y \geq 1, \quad \text{zero otherwise.}$$

(a) Given $n$ samples $\{y_i\}$, find the ML estimator. [Hint: $y^{-\theta} = e^{-\theta \ln y}$]

(b) Find the Cramér-Rao bound on the variance of any estimator for $\theta$. Be sure to state whether it's a lower or upper bound.

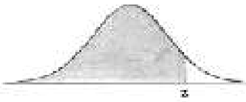
(c) What pdf is the *conjugate prior* for this power-law pdf? (you must show *why*)

Useful formulae and pdfs:

$$
\begin{aligned}
f_{Y_i'}(y) &= \frac{n!}{(i-1)!(n-i)!} F_Y(y)^{i-1}[1-F_Y(y)]^{n-i} f_Y(y) \\
f_W(w) &= \int f_X(x) f_Y(w-x)dx \qquad \text{for} \quad W = X+Y \\
f_W(w) &= \int \frac{1}{|x|} f_X(w/x) f_Y(x)dx \qquad \text{for} \quad W = XY \\
f_W(w) &= \int |x| f_X(x) f_Y(wx)dx \qquad \text{for} \quad W = Y/X \\
\text{poisson } p_X(k;\lambda) &= e^{-\lambda}\frac{\lambda^k}{k!}, \text{ for } k=0,1,2,\ldots, \quad \lambda \geq 0, \quad E(X) = \text{Var}(X) = \lambda \\
\text{gamma } f_Y(y;r,\lambda) &= \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y}, \text{ for } y \geq 0, \qquad E(Y) = \frac{r}{\lambda}, \quad \text{Var}(Y) = \frac{r}{\lambda^2} \\
\text{beta } f_Y(y;r,s) &= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} y^{r-1}(1-y)^{s-1} \text{ for } 0 \leq y \leq 1, \quad E(Y) = \frac{r}{r+s} \\
\text{normal } f_Y(y;\mu,\sigma) &= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, \quad E(Y) = \mu, \quad \text{Var}(Y) = \sigma^2 \\
\text{negative binomial } p_X(k;r,p) &= \binom{k-1}{r-1} p^k (1-p)^{k-r}, \text{ for } k = r, r+1, \ldots, \quad E(X) = \frac{r}{p}, \quad \text{Var}(X) = \frac{r(1-p)}{p^2}
\end{aligned}
$$

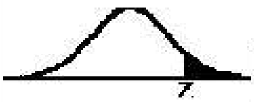
CDF of standard normal follows on next page, and 'far-right tail probabilities' which are $1 - F_Z(z)$ for large $z$ up to 9.5.

Note you can get $F_Z(z)$ for $z < 0$ via $1 - F_Z(-z)$. I also encourage you to skip looking up $F_Z(z)$ values if pushed for time; just write $F_Z$(something).



# Probability Content from -oo to Z

| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |




# Far Right Tail Probabilities

| Z | P{Z to oo} | Z | P{Z to oo} | Z | P{Z to oo} | Z | P{Z to oo} |
|---|---|---|---|---|---|---|---|
| 2.0 | 0.02275 | 3.0 | 0.001350 | 4.0 | 0.00003167 | 5.0 | 2.867 E-7 |
| 2.1 | 0.01786 | 3.1 | 0.0009676 | 4.1 | 0.00002066 | 5.5 | 1.899 E-8 |
| 2.2 | 0.01390 | 3.2 | 0.0006871 | 4.2 | 0.00001335 | 6.0 | 9.866 E-10 |
| 2.3 | 0.01072 | 3.3 | 0.0004834 | 4.3 | 0.00000854 | 6.5 | 4.016 E-11 |
| 2.4 | 0.00820 | 3.4 | 0.0003369 | 4.4 | 0.000005413 | 7.0 | 1.280 E-12 |
| 2.5 | 0.00621 | 3.5 | 0.0002326 | 4.5 | 0.000003398 | 7.5 | 3.191 E-14 |
| 2.6 | 0.004661 | 3.6 | 0.0001591 | 4.6 | 0.000002112 | 8.0 | 6.221 E-16 |
| 2.7 | 0.003467 | 3.7 | 0.0001078 | 4.7 | 0.000001300 | 8.5 | 9.480 E-18 |
| 2.8 | 0.002555 | 3.8 | 0.00007235 | 4.8 | 7.933 E-7 | 9.0 | 1.129 E-19 |
| 2.9 | 0.001866 | 3.9 | 0.00004810 | 4.9 | 4.792 E-7 | 9.5 | 1.049 E-21 |