LECTURE OUTLINE
An Example of Statistical Inference

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Math 50

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Goals

Review Binomial

The Goals of Statistical Inference

Language of Statistical Tests

Non-Bayesian Tests

Bayes' Theorem

Bayesian Tests

Octopus Identification

Two Variety of octopus: the common Madako (90 percent) the rare Nidako (10 percent) as adolescents can only be distinguished by their tendencies to turn maroon or red when disturbed. (Let us pretend.)

	Madako	Nidako
Maroon Frequency	0.2	0.7
Red Frequency	0.8	0.3

Develop a Statistical Test

Hopothesis	Symbol	$\theta = Pr(\text{Turns Maroon})$
Madako	H_0	$\theta_0 = 0.2$
Nidako	H_1	$\theta_1 = 0.7$

We call θ a *parameter*. Develop and analyze a test to determine which parameter value to accept. Call the test δ, where we let $\delta = d_0 = 0$ if our test results in a decision to accept accept H_0 and reject H_1, while let $\delta = d_1 = 1$ if our test results in a decision to accept accept H_1 and reject H_0.

Our Test

Disturb the octopus n times and record the number of times that the octopus turns Maroon, call this number X. Then we some how chose an n_c such that if

$$\delta = \begin{cases} 0 & X < n_c \\ 1 & X \geq n_c \end{cases} .$$

Note: δ depends on fixing two constants, the number of trials n and n_c.

Analyzing a Statistical Test

Fill and interpret the relevance of the following table of probabilities of the following occurrences.

	$\theta = \theta_0$	$\theta = \theta_1$
$\delta = d_0$	$1 - \alpha(\delta)$	$\beta(\delta)$
$\delta = d_1$	$\alpha(\delta)$	$1 - \beta(\delta)$

Analyzing Our Statistical Test

In class we initially choose $n = 1$ and $n_c = 1/2$ and found:

	$\theta = \theta_0$	$\theta = \theta_1$
$\delta = d_0$	$1 - 0.2$	0.3
$\delta = d_1$	0.2	$1 - 0.3$

Analyzing Our Test for $n = 4$ *and* $n_c = 1.5$

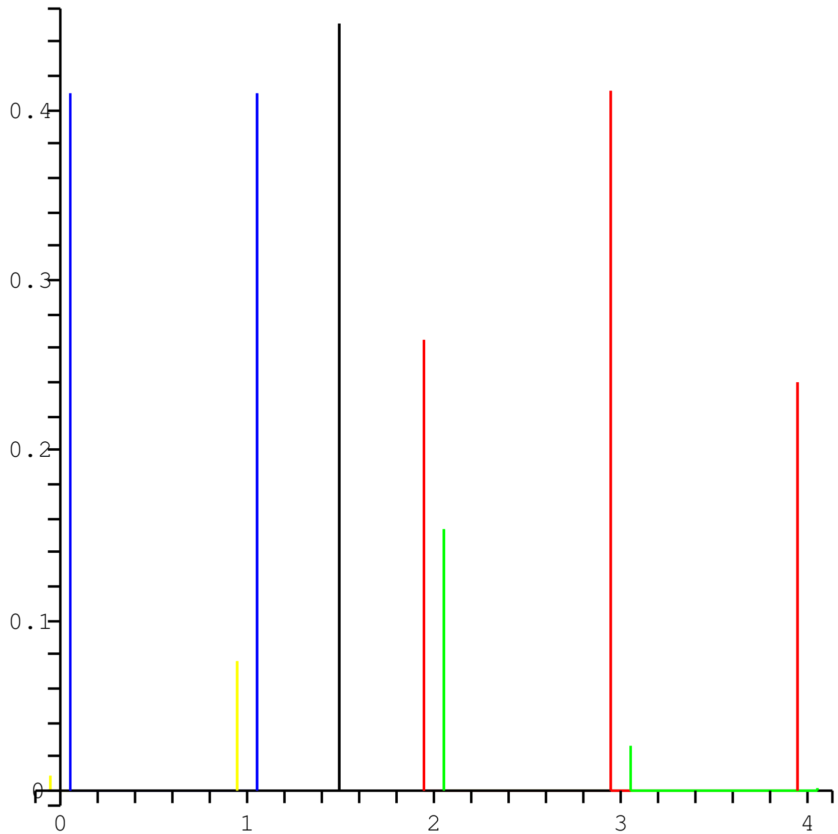
Then we choose $n = 4$ and $n_c = 1.5$. We found:

	$\theta = \theta_0$	$\theta = \theta_1$
$\delta = d_0$	$1 - .1808$	0.0837
$\delta = d_1$	$.1808$	$1 - 0.0837$

Notice risk of incorrectly identifying the octopus if $\theta = \theta_0$ is virtually the same as it was when $n = 1$ and $n_c = 1$.

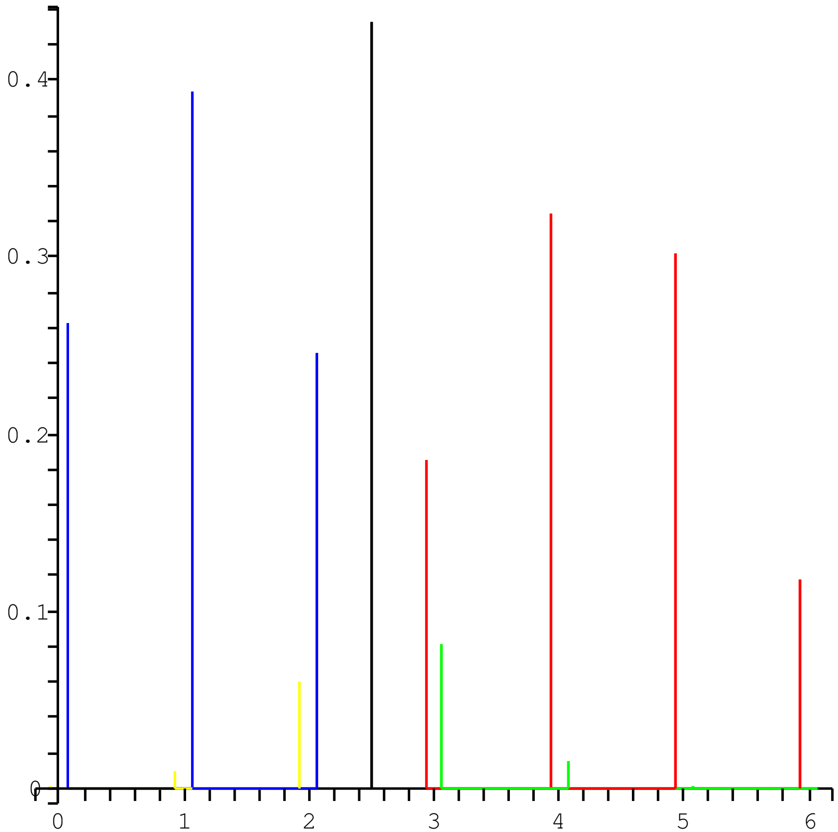
A Picture for $n = 4$ *and* $n_c = 1.5$

Here is a picture of our understanding of out test when $n = 4$ and $n_c = 1.5$.



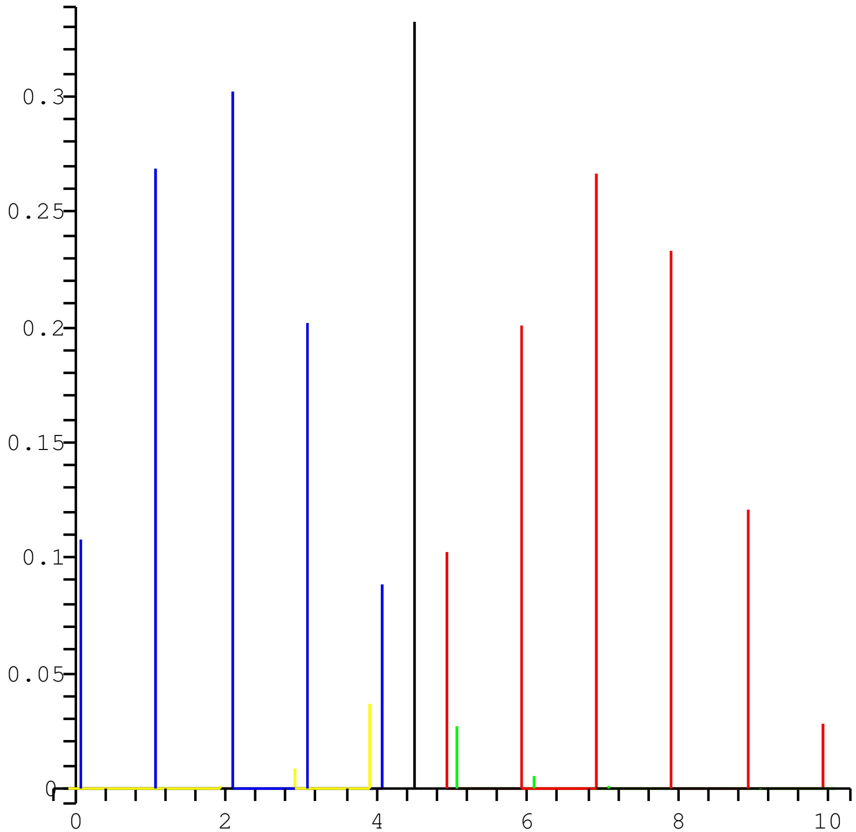
Analyzing Our Test for $n = 6$ *and* $n_c = 2.5$

Out test when $n = 6$ and $n_c = 2.5$ has $\alpha(\delta) = 0.09888$ and $\beta(\delta) = 0.07047$.



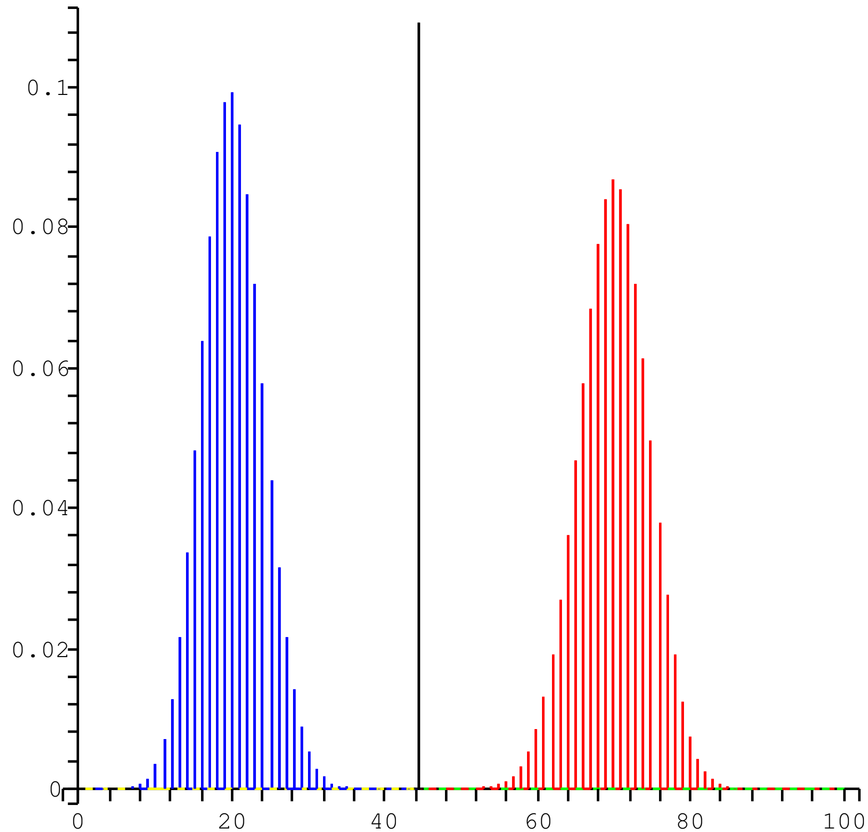
Analyzing Our Test for $n = 10$ *and* $n_c = 4.5$

Out test when $n = 10$ and $n_c = 4.5$ has $\alpha(\delta) = 0.0327$ and $\beta(\delta) = 0.0473$.



Analyzing Our Test for $n = 100$ *and* $n_c = 44.5$

Out test when $n = 100$ and $n_c = 44.5$ has $\alpha(\delta) = 1.4(10^{-8})$ and $\beta(\delta) = 5.8(10^{-8})$.



Power

Explain the significance of the following *Power Function*.

$$\pi(\theta \mid \delta) = \begin{cases} \alpha(\delta) & \theta = \theta_0 \\ 1 - \beta(\delta) & \theta = \theta_1 \end{cases}$$

Some Language

If we call H_0 the *Null Hypothesis* (though this is kind of weird here), then we call H_1 the *Alternate Hypothesis* and give the scenarios in our previous table the following names:

	$\theta = \theta_0$	$\theta = \theta_1$
$\delta = d_0$	No Error	Type 2 Error
$\delta = d_1$	Type 1 Error	No Error

Also we would call the $C = \{n \mid n \geq n_c\}$ the *critical region* for our test. If we run the test and find k maroon, the we call the P-value the $Pr(X \geq k \mid \theta = \theta_0)$.

Our Test In Action

We have ignored something. Suppose we apply our test $n = 6$ and $n_c = 2.5$. We should error on average less than 1 in ten times and we might imagine that this is "acceptable". Now lets go out and collect 30 Nidako using our method.

What percent of our supposed Nidako should we "expect" to be actually be Nidako? In other words, compute

$$Pr(\theta = \theta_1 \mid \delta = 1).$$

Answer: $Pr(\theta = \theta_1 \mid \delta = 1) = 0.51$, using Bayes' Theorem!

Bayes' Theorem

"Baby" Bayes' Theorem: If $Pr(A) > 0$ and $1 > Pr(B) > 0$, then

$$Pr(B \mid A) = \frac{Pr(B)Pr(A \mid B)}{Pr(B)Pr(A \mid B) + Pr(B^c)Pr(A \mid B^c)}.$$

Priors and Posteriors

Let us run the test, and use your data to fill in the following probabilities as we go along:

	θ_0	θ_1
Priors	$Pr(\theta = \theta_0 \mid \text{Before test})$	$Pr(\theta = \theta_1 \mid \text{Before test})$
Posterierors	$Pr(\theta = \theta_0 \mid \text{After test})$	$Pr(\theta = \theta_1 \mid \text{After test})$

Use the information at our disposal, namely

$$Pr(\theta = \theta_i \mid \text{After test}) = Pr(\theta = \theta_i \mid X = x).$$

Priors and Posteriors

We found as we went along that...

$Test$	$Result$	$Pr(\theta = \theta_0 \vert \text{Current Info})$	$Pr(\theta = \theta_1 \vert \text{Current Inf}$
0		0.9	0.1
1	$Maroon$	0.72	0.28
2	Red	0.87	0.13
3	Red	0.95	0.05
4	Red	0.98	0.02
5	Red	0.99	0.01
6	Red	0.997	0.003

Priors and Posteriors

Indeed it was a Madako as indicated by these results. Letting X be the Maroon count, confirm we could have used our original test to find

$$Pr(\theta = \theta_1 | X = 1) = 0.003.$$

Fact: The computation of posterior probabilities can be done in one, step by step, or any method in between.

Develop a Statistical Test

Develop and analyze a test to determine whether an adolescent octopus is a Madako or a Nidako using Bayesian ideas. Call the test δ, where we let $\delta = d_0 = 0$ if our test results in a decision to accept accept H_0 and reject H_1, while let $\delta = d_1 = 1$ if our test results in a decision to accept accept H_1 and reject H_0.

How the Pros Do It: The Loss Function

	$\theta = \theta_0$	$\theta = \theta_1$
$\delta = d_0$	0	ω_0
$\delta = d_1$	ω_1	0

The quantities in the above table are your *losses*. The *Bayes Test* is to let δ be the hypothesis which minimizes the expected loss function using the **posterior probabilities.**

Other Views

Theorem: The Bayes Test tells us that we may accept d_i if

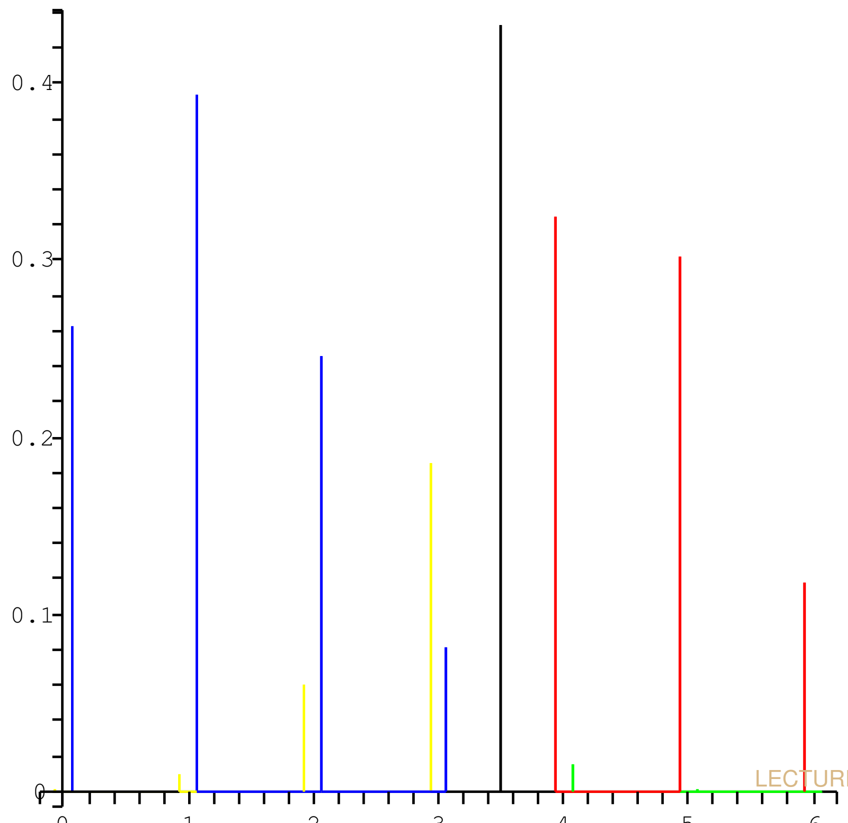
$$Pr(\theta = \theta_i | X = x) \geq \frac{\omega_i}{\omega_i + \omega_j}.$$

Equivalently we may accept d_i if x satisfies

$$Pr(X = x \mid \theta = \theta_i) \geq \frac{\omega_i Pr(\theta = \theta_j)}{\omega_j Pr(\theta = \theta_i)} Pr(X = x \mid \theta = \theta_j).$$

Pictorial View

Here is a picture of our understanding of out test when $n = 6$, $\omega_0 = \omega_1 = 1$. For this test we find $\alpha(\delta) = 0.017$ and $\beta(\delta) = 0.26$.



Pictorial View

Here is a picture of our understanding of out test when $n = 6$, $\omega_0 = 1$ and $\omega_1 = 4$. For this test we find $\alpha(\delta) = .0016$ and $\beta(\delta) = 0.58$. Articulate why.