Worksheet #9: WKB approximation

Consider $\epsilon^2 y'' - xy = 0$, where $\epsilon \ll 1$.

(1) For what domains is ~~it~~ the solution oscillatory? Evanescent (growing/decaying)?

oscillatory for $x < 0$.
evanescent for $x > 0$.

(2) Lets take $x > 1$. Write down $k(x)$.

$k(x) = \sqrt{x}$ so $\int \sqrt{x}\, dx = \frac{2}{3} x^{3/2}$.

(3) Write down the general WKB solution.

$$y(x) = C_1 \frac{1}{x^{1/4}} e^{\frac{1}{\epsilon}\frac{2}{3}x^{3/2}} + C_2 \frac{1}{x^{1/4}} e^{-\frac{1}{\epsilon}\frac{2}{3}x^{3/2}}$$

(4) Find the coefficients with the boundary conditions: $y(1) = 1$, $\lim_{x\to\infty} y(x) = 0$.

$$y(1) = 1 = C_1 e^{2/(3\epsilon)} + C_2 e^{-2/(3\epsilon)}$$

$\lim_{x\to\infty} y(x) = 0 \Rightarrow C_1 = 0$ → otherwise exponential growth.

$\Rightarrow C_2 = e^{2/3\epsilon}$

(5) Rewrite the WKB solution so $x = 1$ is lower limit of action integral.

$$y(x) = C_2 x^{-1/4} e^{-\frac{1}{\epsilon}2/3 x^{3/2}}$$ Plugin C_2

$$= x^{-1/4} e^{-\frac{2}{3\epsilon}(x^{3/2}-1)}$$

$$= x^{-1/4} e^{-2/3\epsilon \int_1^x \sqrt{s}\, ds}$$

where $\int_1^x \sqrt{s}\, ds$ is the action integral w/ lower limit 1.