Worksheet #8: Initial layer

Consider the small mass damped spring equation

$$\epsilon y'' + y' + y = 0 \qquad t > 0$$
$$y(0) = 0 \quad \epsilon y'(0) = 1 \qquad \epsilon \ll 1$$

(1) Write down and solve for the outer layer problem. [Hint: take $\epsilon = 0$] Can you identify the constant?

$y' + y = 0 \Rightarrow y_o = Ce^{-t}$

We will find C when we match.

(2) Rescale the ODE in terms of time taking $\tau = \frac{t}{\delta(\epsilon)}$ and $Y(\tau) = y(t)$.

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{Y'}{\delta} + Y = 0$$

(3) Use dominant balancing to choose a scale $\delta = \epsilon^{\alpha}$ for some α.

Take $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \Rightarrow \delta = O(\varepsilon) \qquad \alpha = 1.$

(4) Rewrite the ODE with this choice of δ.

$\Rightarrow \frac{\varepsilon}{\varepsilon^2} Y'' + \frac{Y'}{\varepsilon} + Y = 0 \rightarrow Y'' + Y' + \varepsilon Y = 0.$

$Y(0) = 0 \qquad Y'(0) = 1$

(5) Find the leading order equation and solve. If possible, find the constants involved.

leading order equation is

$Y'' + Y' = 0. \Rightarrow Y' = Be^{-t} \Rightarrow Y = A + Be^{-\tau}$

$Y(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

$Y'(0) = -B = 1 \Rightarrow B = -1 \quad \Rightarrow A = 1 \quad Y_i(\tau) = 1 - e^{-\tau}$

$\Rightarrow y_i(t) = 1 - e^{-t/\varepsilon}$

(6) Match the two solutions by identifying the constant in part 1. What is the uniform approximation to the solution?

We want $\lim\limits_{t \to 0^+} y_o(t) = \lim\limits_{\tau \to \infty} Y_i(\tau)$

$\Rightarrow \quad C = 1$

So the uniform approximation is

$y_u(t) = y_o(t) + y_i(t) - C = e^{-t} - e^{-t/\varepsilon}$