Worksheet #4: Dimensional analysis II

In this worksheet we explore the fundamental solution for the heat equation (without calculus.) A pulse of energy sized e is released at the origin at time $t = 0$. The medium has heat capacity

(energy per volume per degree) and thermal conductivity K (power per length per degree). The temperature at distance r and time t is u. (We take $u = 0$ everywhere for $t < 0$.

(a) Using the fundamental units energy (E), length (L), time (T), and temperature (Θ), construct the 4×6 dimensional matrix A. (Hint: the fundamental units of K are $EL^{-1}T^{-1}\Theta^{-1}$.

$$\begin{array}{c|c} & \begin{array}{cccccc} e & r & t & u & c & K \end{array} \\ \begin{array}{c} E \\ L \\ T \\ \Theta \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -3 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix} \end{array}$$

(b) Find the $p = 2$ independent dimensionless quantities. (Hint: one does not involve u, the other does not involve r.)

$Kt \Rightarrow$ not in T. $[Kt] = E L^{-1} \Theta^{-1}$ $\Rightarrow \Pi_1 = \dfrac{cr^2}{Kt}$ is unitless.

We want to involve u. $\Rightarrow \Pi_2 = \dfrac{ce^2}{(Kt)^3 u^2}$

(c) What does the Buckingham Pi theorem tell us about these quantities. Use this to find a function u of the everything else.

The Buckingham Pi Thm says $\exists$ a function F st $F(\Pi_1, \Pi_2) = 0$.

$\Rightarrow \exists$ a function g st $\Pi_2 = g(\Pi_1)$

$\Rightarrow \dfrac{ce^2}{(Kt)^3} u^{-2} = g(\Pi_1) \Rightarrow u = \left(\dfrac{c^{1/2} e}{(Kt)^{3/2}}\right) \tilde{g}\left(\dfrac{cr^2}{Kt}\right)$

we will find $\tilde{g}$ later

(d) If $r = 0$, how must u scale with t?

If $r = 0$ $u = \text{constant}\ t^{-3/2} \Rightarrow$ scales $\sim t^{-3/2}$

(e) How does the scaling in part (d) change in a general dimension d. (We had $d = 3$ above. Note, that K has units $ET^{-1}L^{2-d}\Theta^{-1}$ in general.)

$\Rightarrow u \sim t^{-d/2}$