Worksheet #19: Convolution and the Fourier Transform

(1) Let u and v be Schwarz functions. Show that

$$\mathcal{F}(u * v)(\xi) = \hat{u}(\xi)\hat{v}(\xi).$$

$$\int_{-\infty}^{\infty} e^{i\xi x} \int_{-\infty}^{\infty} u(x-y)\,v(y)\,dy\,dx = \int_{-\infty}^{\infty} e^{i\xi(x-y+y)} \int_{-\infty}^{\infty} u(x-y)\,v(y)\,dy\,dx$$

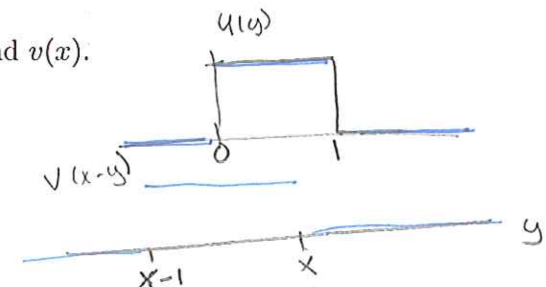
let $z = x-y$
$dz = dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\xi z}\, u(z)\, e^{i\xi y}\, v(y)\,dz\,dy = \hat{u}(\xi)\,\hat{v}(\xi)$$

(2) Work out the convolution of $u(x)$ and $v(x)$.

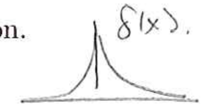
(a) $u(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$v(x) = u(x)$



$$(u*v)(x) = \int_{-\infty}^{\infty} v(x-y)\,u(y)\,dy = \begin{cases} 0 & x<0 \quad x>2 \\ x & 0<x<1 \\ 2-x & 1<x<2 \end{cases}$$

(b) $u(x) =$ any function. $v(x) = \delta(x)$ The delta function.



$$(u*v)(x) = \int_{-\infty}^{\infty} \delta(x-y)\,u(y)\,dy = u(x).$$

(c) $u(x) = v(x) = e^{-\frac{x^2}{2}}$ How wide is the answer compared to the original?

$$(u*v)(x) = \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{2}}\, e^{-y^2/2}\,dy = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + xy - y^2}\,dy$$

Comple the square

$$-(y^2 - xy + \tfrac{x^2}{4}) - \tfrac{x^2}{2} + \tfrac{x^2}{4}$$
$$= -(y - x/2)^2 - \tfrac{x^2}{4}$$

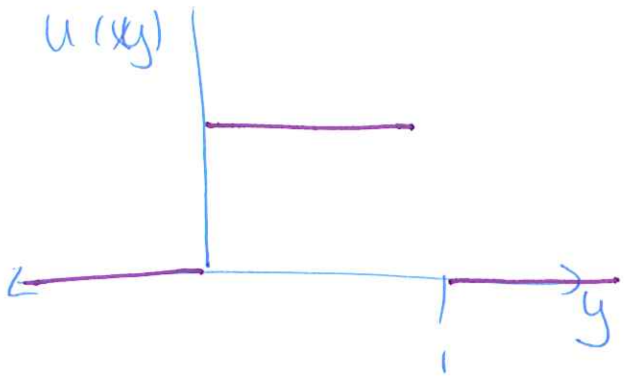
$$= \int_{-\infty}^{\infty} e^{-x^2/4}\, e^{-(y-x/2)^2}\,dy = e^{-x^2/4} \underbrace{\int_{-\infty}^{\infty} e^{-(y-x/2)^2}\,dy}_{=\sqrt{\pi}}$$

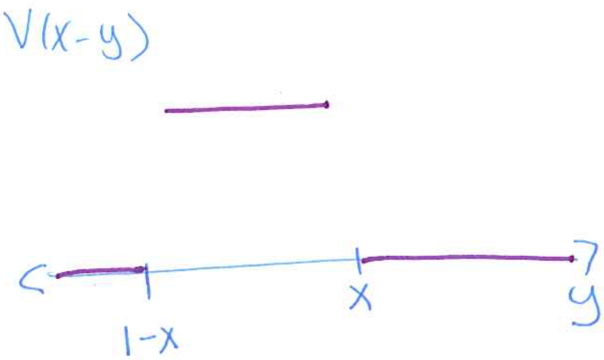
let $z = y - x/2$
$dz = dy$

$$= \sqrt{\pi}\, e^{-x^2/4}.$$

So the gaussian is wider by a factor of $\sqrt{2}$.

$$u(x) = v(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$



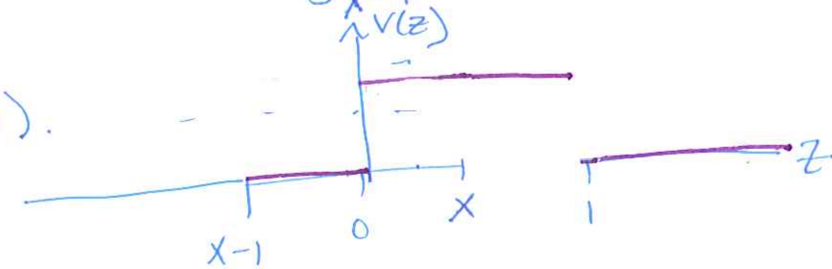


$$(u * v)(x) = \int_{-\infty}^{\infty} v(x-y)\,u(y)\,dy = \int_0^1 v(x-y)\,dy$$

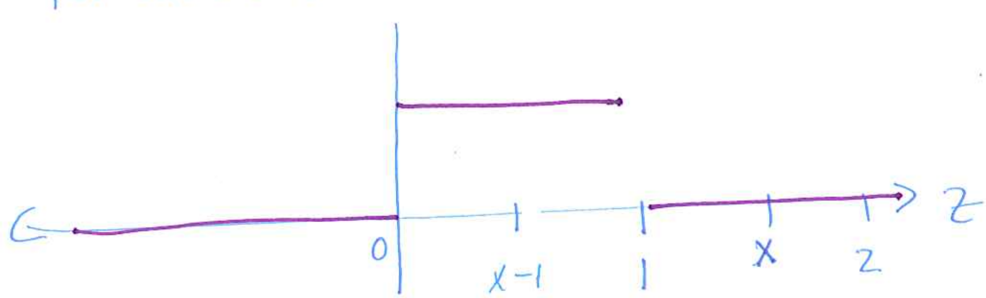
let $z = x - y$
$dz = -dy$

$$= -\int_x^{x-1} v(z)\,dz = \int_{x-1}^{x} v(z)\,dz$$

for $x \in (0,1)$.



for $x \in (1,2)$



So. for $x \in (0,1)$

$$(u*v)(x) = \int_0^x 1\, dz = z\Big|_0^x = x$$

for $x \in (1,2)$

$$(u*v)(x) = \int_{x-1}^1 1\, dz = z\Big|_{x-1}^1 = 1-(x-1) = 2-x.$$