Worksheet #16: Green's functions

Consider the differential operator

$$A = -\frac{d^2}{dx^2}$$

on $[0,1]$ with Dirichlet boundary conditions. We wish to find a Green's function for problems of the form

$$Au = f$$
$$u(0) = u(1) = 0.$$

In this worksheet, we will derive the Green's function.

- Write the general solution to $Au = 0$.

$u = Ax + B.$

- Solve for $u_1(x)$ which only satisfies the left-hand boundary conditions.

$u_1(0) = B = 0 \rightarrow u_1(x) = x$

- Solve for $u_2(x)$ which only satisfies the right-hand boundary conditions.

$u_2(1) = A + B = 0 \rightarrow A = -B$

$u_2(x) = A(1-x) \rightarrow$ let $A = 1$ $\quad u_2(x) = 1 - x$

- Compute the Wronskian. ie $W = \det\left(\begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix}\right)$

$$\begin{vmatrix} x & 1-x \\ 1 & -1 \end{vmatrix} = -x - (1-x) = -x - 1 + x = -1$$

- Write $g(x,\xi)$. $\quad P = -1$

$$g(x,\xi) = \begin{cases} \dfrac{-u_1(x)\,u_2(\xi)}{P(\xi)\,W(\xi)} = \dfrac{-x(\xi-1)}{1} & x < \xi \\[2ex] \dfrac{-u_2(x)\,u_1(\xi)}{P(\xi)\,W(\xi)} = \dfrac{-(1-x)\xi}{1} & x > \xi \end{cases}$$

- ~~Sketch $g(x,\xi)$ in the box $[0,1]^2$~~. Do you notice anything?

The kernel is symmetric.