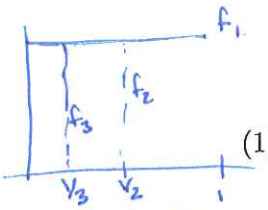


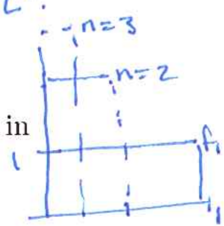
Worksheet #11: L^2 convergence

(1) Is the sequence of functions $f_n(x) = \begin{cases} 1 & x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ convergent on $(0,1)$? If so, in what sense? (pointwise, uniformly, or L^2)

Ptwise: fix x. $\lim_{n\to\infty} f_n(x) = 0$ ✓ yes it converges.

Uniform: $\max_{x\in(0,1)} |f_n(x)| = 1$ $\quad \lim_{n\to\infty} 1 \neq 0$. ⇒ Does not converge uniformly.

L^2: $\int_0^1 (f_n(x))^2 dx = \int_0^{1/n} 1^2 dx = 1/n \quad \lim_{n\to\infty} 1/n = 0$. ⇒ converges in L^2.

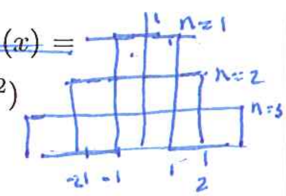


(2) Is the sequence of functions $f_n(x) = \begin{cases} \sqrt{n} & x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ convergent on $(0,1)$? If so, in what sense? (pointwise, uniformly, or L^2)

Ptwise: fix $x \in (0,1)$ $\lim_{n\to\infty} f_n(x) = 0$. ✓ converges ptwise.

Uniformly: $\max_{x\in(0,1)} |f_n(x)| = \sqrt{n} \not\to 0$ as $n\to\infty$ Does not converge uniformly.

L^2: $\int_0^1 (f_n(x))^2 dx = \int_0^{1/n} n\, dx = \frac{n\cancel{x}}{n} = 1 \not\to 0$ as $n\to\infty$ ⇒ Does not converge in L^2



(3) Now consider the unbounded interval $(-\infty, \infty)$. Is the sequence of functions ~~$f_n(x) =$~~ $f_n(x) = \begin{cases} \frac{1}{n} & |x| < n \\ 0 & \text{otherwise} \end{cases}$ convergent? If so, in what sense? (pointwise, uniformly, or L^2)

Ptwise: fix $x \in (-\infty,\infty)$ $\lim_{n\to\infty} (f_n(x)) = 0$. converges ptwise

Uniformly: $\max_{x\in(-\infty,\infty)} |f_n(x)| = 1/n \to 0$ as $n\to\infty$ converges uniformly.

L^2: $\int_{-\infty}^{\infty} (f_n(x))^2 dx = \int_{-n}^{n} 1/n^2 = \frac{1}{n^2}(n+n) \to 0$ as $n\to\infty$ ⇒ converges in L^2.

(4) Modify problem (3) so that the function converges pointwise and uniformly but not in L^2.

If $f_n(x) = \begin{cases} \frac{1}{\sqrt{n}} & \text{for } |x| < n \\ 0 & \text{otherwise.} \end{cases}$,

$f_n(x) \to 0$ ptwise and uniformly

But not in L^2 since

$\int_{-\infty}^{\infty} (f_n(x))^2 dx = \int_{-n}^{n} \frac{1}{n} dx = 2 \not\to 0$ as $n\to\infty$.