Math 46 Spring 2013

Introduction to Applied Mathematics

Second Midterm Exam

Thursday, May 16, 5:00-7:00 PM

Your name (please print): Solutions

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

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For grader use only:

Problem	Points	Score
1	6	
2	10	
3	6	
4	8	
5	6	
6	6	
7	8	
Total	50	

1. [6 points] Find the first two terms in the asymptotic expansion of $I(x) = \int_x^\infty e^{-t^4} dt$ for x large ($x \to \infty$). [Hint: $e^{-t^4} = \frac{1}{-4t^3} \frac{d}{dt}\left(e^{-t^4}\right)$]

$$\int_x^\infty e^{-t^4} dx = \int_x^\infty \underbrace{\frac{1}{-4t^3}}_{u} \underbrace{\frac{d}{dt}\left(e^{-t^4}\right) dt}_{dv}$$

$du = \frac{3}{4} t^{-4}$

$v = e^{-t^4}$

$$= \lim_{s \to \infty} \frac{1}{-4t^3} e^{-t^4} \Big|_x^{s} {}^{\to 0} - \int_x^\infty \frac{3}{4} t^{-4} \cdot e^{-t^4} dt$$

$$= \frac{1}{-4x^3} e^{-x^4} - \int_x^\infty \underbrace{\frac{3}{4} t^{-4} \left(\frac{1}{-4} t^{-3}\right)}_{u} \underbrace{\frac{d}{dt}\left(e^{-t^4}\right) dt}_{dv}$$

$$= \frac{-1}{4x^3} e^{-x^4} - \left[\frac{-3}{16} t^{-7} e^{-t^4} \Big|_x^\infty {}^{\to 0} - \int_x^\infty \frac{+21}{16} t^{-8} e^{-t^4} dt\right]$$

$$= \underbrace{\frac{-1}{4x^3} e^{-x^4} + \frac{3}{16} x^{-7} e^{-x^4}}_{1^{st} \text{ two terms of expansion.}} + \underbrace{\int_x^\infty \frac{21}{16} t^{-8} e^{-t^4} dt}_{\text{remainder } R(x)}$$

[BONUS: prove that the remainder term satisfies the needed condition for an asymptotic expansion]

It is an asymptotic expansion if little-o holds.

ie $\lim_{x \to \infty} \frac{R(x)}{u_2(x)} = 0$

$$\lim_{x \to \infty} \left| \frac{\int_x^\infty \frac{21}{16} t^{-8} e^{-t^4} dt}{3/16\, x^{-7} e^{-x^4}} \right| \le \lim_{x \to \infty} \left| \frac{7 e^{-x^4} \int_x^\infty t^{-8} dt}{x^{-7} e^{-x^4}} \right| = \lim_{x \to \infty} \left| 7 x^7 \frac{t^{-9}}{-9} \Big|_x^\infty \right|$$

$$\overset{3}{=} \lim_{x \to \infty} \frac{7}{9} x^{-2} = 0. \checkmark$$

2. [10 points]

(a) What are the eigenvalues and eigenfunctions for the integral operator

$$[Ku](x) = \int_0^1 xy^3 u(y)dy?$$

The kernel is separable.

$\alpha_1(x) = x \quad \beta_1(x) = x^3$

our goal is to find λ & u st $Ku = \lambda u$. let $c = \int_0^1 \beta_1(y)u(y)dy$

Then $Ac = \lambda c$ where $A = \int_0^1 \beta_1 \alpha_1 \, dy = \int_0^1 y^4 dy = 1/5$

$\Rightarrow \lambda = 1/5$ is an eigenvalue. c = constant = 1 eigenvector.

eigenfunction is $u = x$.

$\Rightarrow \lambda = 0$ is an eigenvalue so many times. eigenfunction are any function $\phi(x)$ st $\langle \phi, x^3 \rangle = 0$.

(b) Solve the integral equation $[Ku](x) - u(x) = x^4$ on $(0, 1)$, or explain why it is not possible. There is a unique solution to this problem. $\lambda = 1$ is not an eigenvalue

$$Ac - c = \int_0^1 x^7 dx = \frac{1}{8} = (1/5 - 1)c = -4/5\,c$$

$$\rightarrow c = -5/32$$

The solution is

$$u(x) = \frac{1}{\lambda}\left(f(x) - \sum_{i=1}^{n} \alpha_i(x) c_i\right) = -1\left(x^4 - \frac{-5}{32}x\right) = \frac{-5}{32}x - x^4$$

(c) Solve the integral equation $[Ku](x) = x$ on $(0, 1)$, or explain why it is not possible.

Yes it is possible. $\lambda = 0$ is an eigenvalue & x is in the span of $\{x\} = \{\alpha_j\}$

$$\frac{1}{5} c = \int_0^1 x^4 dx = x^5/5 = 1/5 \Rightarrow c = 1$$

we need $x \int_0 y^3 u(y) dy = x$ ie $\int_0^1 y^3 u(y) dy = 1$

There are infinitely many such solutions. that satisfy this constraint

(d) Solve the integral equation $[Ku](x) = x^2$ on $(0, 1)$, or explain why it is not possible.

$\{x^2\}$ is not in the span of $\{x\} = \{\alpha_j\}$.

& $\lambda = 0$ is an eigenvalue. Thus there is no solution.

3. [6 points] Consider the integral operator $[Ku](x) = \int_2^{2e} k(x,y)u(y)dy$ with kernel

$$k(x,y) = \begin{cases} 1-\ln\left(\frac{y}{2}\right), & x<y \\ 1-\ln\left(\frac{x}{2}\right), & y<x \end{cases}$$

Convert the eigenvalue problem $Ku = \lambda u$ into a Sturm-Liouville problem on the interval $(2, 2e)$. Do not forget to find homogeneous boundary conditions. [Hint: one will be Dirichlet, one Neumann]

$$\lambda u = Ku = \int_2^x u(y)(1-\ln(x/2))\,dy + \int_x^{2e} u(y)(1-\ln(\tfrac{y}{2}))\,dy$$

Takeaderivative

$$(*) \quad \lambda u' = \int_2^x u(y)\left(\frac{-2}{x}\right) dy + \cancel{u(x)(1-\ln(x/2))} - \cancel{\int_{2e}^x \overset{0}{0}\, dy} - \cancel{u(x)\left(1-\ln\tfrac{x}{2}\right)}$$

$$= \int_2^x u(y)\left(\frac{-2}{x}\right) dy$$

Take another

$$\lambda u'' = \int_2^x u(y)\left(\frac{+2}{x^2}\right) dy + u(x)\left(\frac{-2}{x}\right)$$

$$= \frac{-1}{x}\underbrace{\int_2^x u(y)\left(\frac{-2}{x}\right) dy}_{\lambda u'} + u(x)\left(\frac{-2}{x}\right)$$

$$\lambda u'' + \frac{\lambda u'}{x} + \frac{2}{x}u = 0. \Rightarrow xu'' + u' + \frac{2}{\lambda}u = 0$$

$$\lambda(\underbrace{x}_{P(x)}u')' + \underbrace{\frac{2}{\lambda}}_{q(x)}u = 0$$

BC.

$$u(2e) = \int_2^{2e} u(y)\underbrace{(1-\ln(2e/2))}_{1-\ln e\,=0}dy + \underbrace{\int_{2e}^{2e} ?\,dy}_{=0} = 0$$

from (*) we have

$$u'(2) = \frac{1}{\lambda}\int_2^2 \underbrace{\text{bounded}}\, dy = 0.$$

4. [8 points] Consider the boundary-value problem $-u''(x) + \omega^2 u(x) = f(x)$ for $\omega^2 > 0$ (a fixed constant) on the interval $x \in [0, 1]$ with Dirichlet boundary conditions $u(0) = u(1) = 0$.

(a) Can a Greens function exist for this problem? (Why?)

Is zero an eigen value of $Lu = \left(-\frac{\partial^2}{\partial x^2} + \omega^2\right) u$?

Homogeneous solution $u(x) = C_1 e^{-\omega x} + C_2 e^{\omega x}$

$u(0) = C_1 + C_2 = 0 \rightarrow C_1 = -C_2$

$u(1) = C_1(e^{-\omega} - e^{+\omega}) = 0 \rightarrow C_1 = 0.$

$\rightarrow$ only trivial soln. $\Rightarrow \lambda = 0$ is not an eigenvalue

Thus a green's function exist.

(b) If the Greens function can exist, find it. Otherwise solve the problem for general $f(x)$ another way.

The greens function is given by

$$g(x,\xi) = \begin{cases} -\dfrac{u_1(x)\,u_2(\xi)}{p(\xi)\,W(\xi)} & x<\xi \\[2ex] -\dfrac{u_2(x)\,u_1(\xi)}{p(\xi)\,W(\xi)} & x>\xi \end{cases}$$

$x<\xi$: $= +\dfrac{[e^{-\omega x} - e^{\omega x}](e^{2\omega - \omega\xi} + e^{\omega\xi})}{2\omega(1 + e^{2\omega\xi})}$

$x>\xi$: $= \dfrac{[e^{-\omega\xi} - e^{\omega\xi}](e^{2\omega - \omega x} + e^{\omega x})}{2\omega(1 + e^{2\omega\xi})}$

$P(x) = -1$

u_1 satisfies left bc.

$u_1(x) = C_1 e^{-\omega x} + C_2 e^{\omega x}$ let $C_1 = 1$

$u_1(0) = C_1 + C_2 = 0 \rightarrow C_2 = -C_1$

$u_1(x) = e^{-\omega x} - e^{\omega x}$

u_2 satisfies right bc

$u_2(1) = C_1 e^{-\omega} + C_2 e^{\omega} = 0 \Rightarrow C_1 e^{-\omega} = -C_2 e^{\omega}$

$C_1 = C_2 e^{2\omega}$ let $C_2 = 1$

$u_2(x) = e^{2\omega - \omega x} + e^{\omega x}$

$$W(x) = \begin{vmatrix} e^{-\omega x} - e^{\omega x} & e^{2\omega - \omega x} + e^{\omega x} \\ -\omega e^{-\omega x} - \omega e^{\omega x} & -\omega e^{2\omega - \omega x} + \omega e^{\omega x} \end{vmatrix} = (e^{-\omega x} - e^{\omega x})(\omega(-e^{2\omega - \omega x} + e^{\omega x}) - (-\omega)(e^{-\omega x} - e^{\omega x})(e^{2\omega - \omega x} + e^{\omega x})$$

$$= \omega(-e^{2\omega - 2\omega x} + 1 - e^{2\omega x}) + \omega(e^{2\omega - 2\omega x} + 1 - e^{2\omega x}) = \omega 2(1 + e^{2\omega x})$$

5. [6 points] Find the first 2 non-zero terms in the Neumann series solution of the following Volterra integral equation.

$$u(x) = e^x + \int_0^x e^{y-x} u(y)dy$$

$$= f + \boxed{K(u)}(x) \; u$$

Use Newman series

$u_0 = 0 \qquad u_{n+1} = f + Ku_n$

$u_1 = e^x$

$u_2 = e^x + e^{-x} \int_0^x e^{2y} dy = e^x + \frac{e^{-x}}{2}(e^{2x} - 1) = e^x + \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

$= \frac{3}{2}e^x - \frac{1}{2}e^{-x}$

6. [6 points] What can be deduced about the sign of the eigenvalues of

$$y'' + x^3 y = \lambda y$$

with boundary conditions $y(-1) = y(0) = 0$?

We should use an energy method.

multiply by y & integrate over $[-1, 0]$

$$\int_{-1}^{0} \underbrace{y}_{u}\,\underbrace{y''dx}_{dv} + \int_{-1}^{0} x^3 y^2\,dx = \lambda \int_{-1}^{0} y^2\,dx$$

$$\underbrace{yy'\Big|_{-1}^{0}}_{\substack{= \\ 0 \\ \text{by BC.}}} - \underbrace{\int_{-1}^{0} (y')^2\,dx}_{\leq 0} + \underbrace{\int_{-1}^{0} \underset{<0}{x^3} y^2\,dx}_{\leq 0} = \lambda \underbrace{\int_{-1}^{0} y^2\,dx}_{>0}$$

$\Rightarrow \lambda \leq 0.$

7. [8 points]

(a) Determine if there is a Green's function associated with the operator $Lu = u''+9u$, $0 < x < \pi$, with $u(0) = u(\pi) = 0$.

Is $\lambda=0$ an eigenvalue for Lu.

homogeneous solution is

$u(x) = c_1 \cos(3x) + c_2 \sin(3x)$

$u(0) = c_1 = 0$, $\quad u(\pi) = c_2 \sin(3\pi) = 0$

$\Rightarrow$ 0 is an eigenvalue w/ eigenfunction

$\phi(x) = \sin(3x)$

$\Rightarrow$ no Green's function.

(b) Assuming $f(x) \in L^2([0,\pi])$, find all solutions to the boundary value problem

$$u'' + 9u = f(x), \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.$$

The only way we can have a solution is if $f(x)$ is orthogonal to $\sin(3x)$.

since $f(x) \in L^2$ & $\{\sin(nx)\}_{n=1}^{\infty}$ are complete in L^2

$f(x) = \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} f_n \sin(nx)$ where $f_n = \dfrac{\int_0^\pi f(x)\sin(nx)\,dx}{\int_0^\pi \sin^2(nx)\,dx}$.

We seek $u(x) \in L^2([0,\pi])$.

$u(x) = \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} u_n \sin(nx)$. Plug into problem.

$$\sum_{\substack{n=1 \\ n\neq 3}}^{\infty} (-n^2+9)u_n \sin(nx) = \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} f_n \sin(nx).$$

$$\Rightarrow u_n = \frac{f_n}{9-n^2} \quad n \neq 3.$$

All solutions are given by $u(x) = \underbrace{c\sin(3x)}_{u_{\text{homogeneous}}} + \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} \frac{f_n}{9-n^2}\sin(nx)$