MATH 46 WORKSHEET : WKB for eigenvalues. 4/20/07 Barnett

Transform the more usual eigenproblem

$$-y'' = \lambda q(x) y \qquad y(0) = y(1) = 0$$

into our form $\varepsilon^2 y'' + k^2(x) y = 0$. , ie find ε, $k(x)$:

Will WKB apply to large or small λ?

Use WKB to give approx. to n^{th} eigenvalue λ_n for

$$-\frac{1}{(2-x^2)^2} y'' = \lambda y \qquad y(0) = y(1) = 0$$

If time, write the WKB eigenfunctions :

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~ SOLUTIONS ~

Transform the more usual eigenproblem

$$-y'' = \lambda q(x) y \qquad y(0) = y(1) = 0$$

into our form

$$\varepsilon^2 y'' + k^2(x) y = 0. \quad \text{, ie find } \varepsilon, k(x):$$

$$\frac{1}{\lambda} y'' + q(x) y = 0 \quad \text{so} \quad \varepsilon = \frac{1}{\sqrt{\lambda}}, \ k(x) = \sqrt{q(x)}$$

Will WKB apply to large or small λ? large λ, ie small ε.

Use WKB to give approx. to n^{th} eigenvalue λ_n for

$$-\frac{1}{(2-x^2)^2} y'' = \lambda y \qquad y(0) = y(1) = 0$$

rearrange

$$y'' + \lambda (2-x^2)^2 y = 0$$

$$\varepsilon^2 y'' + (2-x^2)^2 y = 0 \qquad \text{with } \lambda = \frac{1}{\varepsilon^2}$$

$$\text{so } k(x) = 2 - x^2$$

$$\int_0^1 k(x)\,dx = \int_0^1 (2-x^2)\,dx = 2 - \tfrac{1}{3}1^3 = \frac{5}{3}$$

WKB eigenvalues

$$\varepsilon_n \approx \frac{\int_0^1 k(x)\,dx}{\pi n} \quad \text{so} \quad \lambda_n \approx \left(\frac{\pi n}{\int_0^1 k(x)\,dx}\right)^2 = \left(\frac{3\pi n}{5}\right)^2$$

If time, write the WKB eigenfunctions :

$$y_n(x) = \frac{1}{\sqrt{k(x)}} \sin\left(\frac{1}{\varepsilon_n}\int_0^x k(s)\,ds\right) = \frac{1}{\sqrt{2-x^2}} \sin\left[\pi n \frac{\int_0^x k(s)\,ds}{\int_0^1 k(s)\,ds}\right]$$

$$= \frac{1}{\sqrt{2-x^2}} \sin\left[\frac{3\pi n(2x - \frac{1}{3}x^3)}{5}\right]$$