MATH 46 WORKSHEET : Reality of eigenvalues

4/28/08
Barnett.

Consider SLP with $p \equiv 1$, and $q(x)$ real :

$$\boxed{-y'' + q(x)y = \lambda y}$$

(Schrödinger-type ODE)

We'll use Dirichlet BCs $y(a) = y(b) = 0$

Write out ODE mult. by $\bar{y}(x)$:

Write out $\overline{\text{ODE}}$ mult. by $y(x)$:

meaning, ODE for $\bar{y}(x)$

Subtract the two (should be a cancellation) :

Integrate $\int_a^b dx$ everything & use 'by parts' to move a term to bdry :

Use BCs to kill the bdry term :

What is the sign of $\int_a^b y\bar{y}\,dx$? [Hint. $(x+iy)(x-iy) = \cdots$]

Conclude something about $\lambda - \bar{\lambda}$, hence reality of λ.

- What other BCs would this work for ? Neumann?

Periodic $\begin{cases} y(b) = y(a) \\ y'(b) = y'(a) \end{cases}$?

Mixed $\begin{cases} y'(a) = \alpha y(a) \\ y'(b) = \beta y(b) \end{cases}$?

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~ SOLUTIONS ~

Consider SLP with $p \equiv 1$, and $q(x)$ <u>real</u> :

$$\boxed{-y'' + q(x)y = \lambda y}$$

(Schrödinger-type ODE)

We'll use Dirichlet BCs $y(a) = y(b) = 0$

Write out ODE mult. by $\bar{y}(x)$: $-\bar{y}y'' + q\bar{y}y = \lambda\bar{y}y$

Write out $\overline{\text{ODE}}$ mult. by $y(x)$: $-y\bar{y}'' + \bar{q}\bar{y}y = \bar{\lambda}\bar{y}y$ but note q real so $\bar{q} = q$.

meaning, ODE for $\bar{y}(x)$:

$\overline{-y'' + qy = \lambda y} \Rightarrow -\bar{y}'' + \bar{q}\bar{y} = \bar{\lambda}\bar{y}$

$\bar{q} = q$ since q real.

cancels.

Subtract the two (should be a cancellation) : $-\bar{y}y'' + y\bar{y}'' = (\lambda - \bar{\lambda})\bar{y}y$

Integrate $\int_a^b dx$ everything & use 'by parts' to move a term to b'dry: $\int_a^b \cancel{\bar{y}'y' - y'\bar{y}'}\, dx + [-\bar{y}y' + y\bar{y}']_a^b = (\lambda - \bar{\lambda})\int_a^b \bar{y}y\, dx$

cancel

Use BCs to kill the b'dry term : since $y(a) = 0$, $y(b) = 0$ all 4 terms killed.

What is the sign of $\int_a^b y\bar{y}\, dx$? [Hint. $(x+iy)(x-iy) = x^2 + y^2$] ≥ 0

Positive (not zero)

$= \int_a^b (\text{Re}\, y)^2 + (\text{Im}\, y)^2\, dx \geq 0$ Cannot be zero since then $y(x) \equiv 0$, not an eigenfunction.

Conclude something about $\lambda - \bar{\lambda}$, hence reality of λ.

so $0 = (\lambda - \bar{\lambda})$ (positive number) $\Rightarrow \lambda - \bar{\lambda} = 0$ ie $\lambda = \bar{\lambda}$, λ real.

- What other homogeneous BCs would this work for ? Neumann? $y'(a) = y'(b) = 0$ yes.

Periodic $\begin{cases} y(b) = y(a) \\ y'(b) = y'(a) \end{cases}$?

yes since $[\bar{y}y']_a^b = 0$ and $[y\bar{y}']_a^b = 0$ separately.

Mixed $\begin{cases} y'(a) = \alpha y(a) \\ y'(b) = \beta y(b) \end{cases}$?

b'dry term: $[\cdots]_a^b = -\bar{y}\beta y + y\bar{\beta}\bar{y} + \bar{y}\alpha y - y\bar{\alpha}\bar{y}$

Yes only if α, β real (need not be equal)